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The proportion of time spent by a random walk in a subset of the set of integers

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Abstract : $\{X_n\}$, $n = 1, 2......$is a sequence of independent identically distributed random variables such that $P(X_1 = 0) < 1$ and d is the greatest positive integer such that

$$P\{X_1 \in (0, \pm d, \pm 2d, ...)\} = 1.$$

If A is, a subset of the set of integers, of the form $\{kr \mid k = 0, \pm 1, \pm 2, ...\}$ then it is shown that the proportion of time spent by the random walk defined in terms of $\{X_n\}$ in the point set A converges almost surely to b/r as n tends to infinity where b=*g.c.d.* (d,r).

1. Introduction

The problem of "occupation time" has been considered by many authors. Both Feller[4] and Parzen[7] consider the occupation time for a random walk or a Markov chain.

Consider a Markov chain $\{X_n\}$; $n = 0, 1, 2,$............................

For any state k and $n = 1, 2,......$ define $N_n(k)$ to be the number of times the state k is occupied in the first n transitions. $N_n(k)$ is called the occupation time of the state k in the first n transitions.

Parzen[7] extends the above concept to consider the total occupation time of the state k, defined as

$$N_\infty(k) = \lim_{n\to\infty} N_n(k)$$

and the average occupation time $\lim_{n\to\infty} \frac{N_n(k)}{n}$ which for a positive recurrent Markov chain is shown to be equal to the component of the stationary initial distribution of the chain corresponding to the state k.

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It has been shown that if X_1, X_2, ... have lattice distance one and $P(S_n = o\ i.o.) = 1$, where $S_n = X_1 + X_2 + \ldots + X_n$, then for any two states i, j

$$\frac{N_n(i)}{N_n(j)} \xrightarrow{a.s.} 1 \text{ as } n \longrightarrow \infty$$

If X_0, X_1, ...is a Markov chain starting from an initial state i such that i is recurrent, then

$$\frac{N_n(j)}{N_n(k)} \xrightarrow{a.s.} \frac{\pi(j)}{\pi(k)} \text{ as } n \longrightarrow \infty$$

where $\pi(j)$ is the expected number of visits to state j before return to i and $\pi(i) = 1$.

Suppose X_1, X_2,......... are distributed on the lattice $L_d = \{nd \mid n = 0, \pm 1, \pm 2, \ldots\}$ we derive results concerning the proportion of time spent by the random walk in the point set A where A is of the form $A = \{kr / k = 0, \pm 1, \pm 2, \ldots\ldots\}$ r being a fixed integer.

Let X_1, X_2, ...be a sequence of independent, identically distributed random variables taking only integral values with $P(X_1 = 0) < 1$. Let d be the greatest positive integer such that $P\{X_1 \in (\ldots\ldots, -2d, -d, 0, d, 2d, \ldots\ldots)\} = 1$.

Let $S_n = X_1 + X_2 + \ldots\ldots + X_n$, $S_0 = o$. It can be easily proved that $\{S_n\}$ is a Markov chain with stationary transition probabilities and $P(S_{n+1} = d_{n+1} \mid S_n = d_n) = P(X_{n+1} = d_{n+1} - d_n) = P(X_1 = d_{n+1} - d)$.

$$\text{Let } \tau_n = \frac{(\text{Number of } k \text{ such that } 1 \leqslant k \leqslant n \text{ and } S_k \in A)}{n}$$

The main result of this paper is given in the next theorem.

Theorem 1.1.

Suppose the independent, identically distributed random variables X_1, X_2,............are such that $P(X_1 = 0) < 1$, d is the greatest positive integer such that $P\{X_1 \in (0, \pm d, \pm 2d, \ldots\ldots)\} = 1$, τ_n and the set A are as defined above, then $\tau_n \xrightarrow{a.s.} \frac{b}{r}$ as $n \longrightarrow \infty$ (1)

where $b = g.c.d.\ (d.\ r)$.

In order to prove theorem 1.1 we need the number of k such that $1 \leqslant k \leqslant n$ and $S_k \in A$. For this we define the following stochastic process $\{Y_r\}$.

Let $Y_n = s$ if and only if $S_n \equiv s \pmod{r}$, $0 \leqslant s \leqslant r-1$ (2)

Then the number of times $S \in A$ in the first n transitions is given by the number of Y_k's such that $Y_k = 0$, $1 \leqslant k \leqslant n$.

Let $N_n(o)$ = Number of k such that $1 \leqslant k \leqslant n$ and $Y_k = 0$.

Then in order to prove (1) it is sufficient if we can prove that

$$\frac{N_n(o)}{n} \xrightarrow{a.s.} \frac{b}{r} \quad \text{as } n \longrightarrow \infty \tag{3}$$

2. Some properties of the process $\{Y_r\}$

The stochastic process $\{Y_r\}$ is defined in terms of the random walk S_n.

In a simple random walk each state can be reached from every other state. Hence if the random walk is simple then the state space of the stochastic process $\{Y_r\}$ is the set of integers $\{o, 1, 2, \ldots\ldots\ldots, r-1\}$.

If the random walk is arbitrary then the state space of the stochastic process $\{Y_r\}$ is a subset of the set of integers $\{o, 1, 2, \ldots\ldots r-1\}$.

Since $b = g.c.d.(d,r)$ there exists integers k_1, k_2 such that $d = k_1 b$, $r = k_2 b$ and $g.c.d.(k_1, k_2) = 1$.

Now $S_n = kd$ where $k = o, \pm 1, \pm 2 \ldots\ldots,$

$S_n = k k_1 b$

Therefore $Y_n = s$ if and only if $k k_1 b = s \pmod{r}$, $o \leqslant s \leqslant r-1$.

(i.e.) $Y_n = s$ if and only if $k k_1 b = mr + s$, where m is an integer.

Thus $s = k k_1 b - mr$

$= (k k_1 - m k_2) b$

Now $o \leqslant s \leqslant r-1$

Hence $o \leqslant k k_1 - m k_2 \leqslant \frac{r}{b} - \frac{1}{b}$

But $k k_1 - m k_2$ is an integer.

Therefore the possible values of s are $o, b, 2b \ldots\ldots, \left(\frac{r}{b} - 1\right) b$.

Hence if the random walk is arbitrary then the state space of the stochastic process $\{Y_r\}$ is the set of integers

$$I_1 = \{o,\ b,\ 2b, \ldots\ldots,\ \left(\frac{r}{b} - 1\right) b\}.$$

Proposition 2.1

$\{Y_r\}$ is a Markov chain with stationary transition probabilities.

Proof:

Let $y_0, y_1, \ldots\ldots, y_{m+1}$ be any set of values from the set of integers I_1, such that

$$P(Y_0 = y_0, Y_1 = y_1, \ldots\ldots, Y_m = y_m) > 0.$$

Then $P(Y_{m+1} = y_{m+1} \mid Y_m = y_m, \ldots\ldots Y_0 = y_0)$

$$= \frac{\sum\limits_{k_0}\sum\limits_{k_1}\ldots\sum\limits_{k_m}\{\sum\limits_{k_{m+1}} P(S_{m+1} = y_{m+1} + k_{m+1} r,\ S_m = y_m + k_m r, \ldots\ldots, S_0 = y_0 + k_0 r)\}}{\sum\limits_{k_0}\sum\limits_{k_1}\ldots\ldots\sum\limits_{k_m} P(S_m = y_m + k_m r, \ldots\ldots, S_0 = y_0 + k_0 r)} \quad (4)$$

Now for fixed $k_0, k_1, \ldots\ldots, k_m$ we have

$$\frac{\sum\limits_{k_{m+1}} P(S_{m+1} = y_{m+1} + k_{m+1} r,\ S_m = y_m + k_m r, \ldots\ldots, S_0 = y_0 + k_0 r)}{P(S_m = y_m + k_m r, \ldots\ldots, S_0 = y_0 + k_0 r)}$$

$$= \sum_{k_{m+1}} P(S_{m+1} = y_{m+1} + k_{m+1} r \mid S_m = y_m + k_m r, \ldots\ldots, S_0 = y_0 + k_0 r)$$

$$= \sum_{k_{m+1}} P(S_{m+1} = y_{m+1} + k_{m+1} r \mid S_m = y_m + k_m r)$$

$$= \sum_{k_{m+1} = -\infty}^{\infty} P(X_{m+1} = y_{m+1} - y_m + (k_{m+1} - k_m) r)$$

$$= \sum_{j = -\infty}^{\infty} P(X_1 = y_{m+1} - y_m + jr)$$

$$= P(X_1 \equiv y_{m+1} - y_m \pmod{r})$$

So (4) becomes

$$\frac{\underset{k_0}{\Sigma}\underset{k_1}{\Sigma}\cdots\underset{k_m}{\Sigma} P\,(X_1 \equiv y_{m+1} - y_m \ (\text{mod r}))\ P\,(S_m = y_m + k_m r, \ldots, S_0 = y_0 + k_0 \text{r})}{\underset{k_0}{\Sigma}\underset{k_1}{\Sigma}\cdots\underset{k_m}{\Sigma} P\,(S_m = y_m + k_m r, \ldots\ldots\ldots, S_0 = y_0 + k_0\, r)}$$

$$= \quad P(X_1 \equiv y_{m+1} - y_m \ (\text{mod } r)).$$

As before it can be shown that $P\,(Y_{m+1} = y_{m+1} \mid Y_m = y_m)$
$= \quad P\,(X_1 \equiv y_{m+1} - y_m \ (\text{mod } r))$.

Thus we have

$P\ (Y_{m+1} = y_{m+1} \mid Y_m = y_m, \ldots\ldots, Y_0 = y_0) = P\,(Y_{m+1} = y_{m+1} \mid Y_m = y_m)$
and hence the proposition.

The state space of the Markov chain $\{Y_r\}$ is finite and every state of the Markov chain $\{Y_r\}$ can be reached from every other state.

Thus $\{Y_r\}$ is a finite irreducible Markov chain.

The transition probabilities of the Markov chain $\{Y_r\}$ are given by

$$p_{ij} = P\ (Y_{m+1} = j \mid Y_m = i)$$
$$= P\,(X_1 = j - i \ (\text{mod } r))$$
$$= \phi\,(j - i \ (\text{mod r})), \text{ say,}$$

Now $\sum_{j \in I_1} p_{ij} = 1$ for all i.

That is $\sum_{j} \phi\,(j - i\ (\text{mod } r)) = 1$ for all i which implies that $\sum_{i} \phi\,(j - i\ (\text{mod r}))$
$= 1$ for all j.

That is $\sum_{i \in I_1} p_{ij} = 1$ for all j.

Thus the transition probability matrix of the Markov chain $\{Y_r\}$ is doubly stochastic.

Proposition 2.2

All states of the Markov chain $\{Y_r\}$ are positive recurrent.

Proof :

$\{Y_r\}$ is a finite irreducible Markov chain.

In a finite Markov chain at least one state is recurrent and any recurrent state is positive recurrent.

Furthermore in an irreducible chain all the states are of the same type.

Hence all the states of the Markov chain $\{Y_r\}$ are positive recurrent.

Define R_k as the time of the $k-$th return of the Markov chain $\{Y_r\}$ to the state o and the time between returns as

$$T_1 = R_1, \quad T_k = R_k - R_{k-1}, \quad (k > 1).$$

$T_1, T_2, \ldots\ldots$ are called the recurrence times for the state o.

Since $\{Y_r\}$ is a finite irreducible Markov chain $T_1, T_2, \ldots\ldots$are independent, identically distributed random variables. Also since all the states of the Markov chain are positive recurrent $E(T_1) < \infty$.

3. On the proposition of time the process $\{Y_r\}$ spends in the state o.

Theorem 3.1

Let $Y_0, Y_1, \ldots\ldots$ be the stochastic process defined by (2). Let $N_n\ (o)$ be the number of visits of the $Y_1, Y_2, \ldots\ldots Y_n$ to the state o. Then

$$\frac{N_n\ (o)}{n} \xrightarrow{a.s.} \frac{b}{r} \quad \text{as } n \longrightarrow \infty \tag{5}$$

Proof :

As defined earlier let $T_1, T_2, \ldots\ldots$ be the recurrence times for the state o.

The stochastic process defined by the equation (2) is a Markov chain with all the states positive recurrent and the recurrence times $T_1, T_2, \ldots\ldots$ are independent, identically distributed random variables with $E\ (T_1) < \infty$.

Therefore by applying the Strong Law of Large Numbers we get

$$\frac{T_1 + T_2 + \ldots\ldots + T_k}{k} \xrightarrow{a.s.} E\ (T_1)$$

That is $$\frac{k}{T_1 + T_2 + \ldots\ldots + T_k} \xrightarrow{a.s.} [E\ (T_1)]^{-1} \tag{6}$$

Now $\dfrac{k}{T_1 + T_2 + \ldots\ldots + T_k}$ is a subsequence of $\dfrac{N_n\ (o)}{n}$.

Thus (6) gives the convergence of $\dfrac{N_n\,(o)}{n}$ along a random subsequence.

To obtain the convergence of $\dfrac{N_n\ (o)}{n}$ over the full sequence we proceed as follows:

Let $T_1 + T_2 + \ldots\ldots + T_k \leqslant n < T_1 + T_2 + \ldots\ldots + T_{k+1}$

Then $\dfrac{N_n\ (o)}{n} = \dfrac{k}{n} \leqslant \dfrac{k}{T_1 + T_2 + \ldots\ldots\ldots T_k}$

and at the same time

$$\frac{N_n\ (o)}{n} = \frac{k}{n} > \frac{k}{T_1 + T_2 \ldots\ldots\ldots + T_{k+1}}$$

Thus we have

$$\frac{k}{T_1 + T_2 + \ldots\ldots\ldots + T_{k+1}} < \frac{N_n\,(o)}{n} \leqslant \frac{k}{T_1 + T_2 + \ldots\ldots + T_k} \tag{7}$$

By (6) $\dfrac{k}{T_1 + T_2 + \ldots\ldots\ldots + T_k} \xrightarrow{a.s.} \dfrac{1}{E\ (T_1)}$

and also $\dfrac{k}{T_1 + T_2 + \ldots\ldots\ldots + T_{k+1}} = \dfrac{k+1}{T_1 + T_2 + \ldots\ldots\ldots + T_{k+1}} \cdot \dfrac{k}{k+1}$

$$\xrightarrow{a.s.} \frac{1}{E\ (T_1)}$$

Hence from (7) we get the convergence of $\dfrac{N_n\ (o)}{n}$ over the full sequence.

So $\dfrac{N_n\ (o)}{n} \xrightarrow{a.s.} \dfrac{1}{E\ (T_1)}$

The proof of the theorem 3.1 is complete if we show that $E\ (T_1) = \dfrac{r}{b}$.

4. Stationary Initial Distribution for the Markov Chain $\{Y_r\}$

The Markov chain $\{Y_r\}$ starts from the state o and the state o is positive recurrent.

Hence the Markov chain $\{Y_r\}$ has a stationary initial distribution $\pi(i)$, $i \in I_1$. [Proposition 7.34,[1]]

$$\text{Furthermore} \quad \pi(k) = \sum_j \pi(j)\, p_{jk}, \quad j, k, \in I_1 \tag{8}$$

$$\pi(k) \geqq o$$

$$\text{and} \quad \sum_k \pi(k) = 1 \tag{9}$$

Also if T_1 is the first recurrence time for the state i, then

$$E(T_1) = \frac{1}{\pi(i)} \tag{10}$$

Theorem 4.1

If the transition probabilities p_{ij} are such that $\sum_{i \in I} p_{ij} = 1$ for all $j \in I$ then the unique (to within a constant multiple) solution to the system of equations

$$\pi(k) = \sum_j \pi(j)\, p_{jk} \tag{11}$$

is constant.

Proof :

It is clear that $\pi(j) = c$ for all j where c is a constant, is a solution to (11).

To prove the uniqueness of the solution we proceed as follows:

$$\text{Suppose} \quad s(k) = \sum_j s(j)\, p_{jk} \text{ for all } k \tag{12}$$

Let $\alpha = \min_k s(k)$, say $\alpha = s(k_o)$

We claim that $s(j) = \alpha$ for all j.

For suppose $s(j_o) > \alpha$ for some j.

We can find an n such that $p^{(n)}_{j_o k_o} > 0$.

By induction we can show that if (12) is true then

$$s(k) = \sum_j s(j)\, p^{(n)}_{jk}$$

So we obtain $\alpha = s(k_0) = \sum_j s(j)\, p^{(n)}_{jk_o}$

$$= \sum_{j \neq j_o} s(j)\, p^{(n)}_{jk_o} + s(j_o)\, p^{(n)}_{j_o k_a}$$

$$> \alpha \sum_{j \neq j_0} p^{(n)}_{jk_o} + \alpha\, p^{(n)}_{j_o k_o}$$

$$= \alpha,$$

which is a contradiction.

Hence $s(j) = a$ for all j.

Now with the help of theorem 4.1 we can find $E(T_1)$.

We know that for the Markov chain $\{Y_r\}$

$$\sum_{i \in I_1} p_{ij} = 1 \text{ for all } j \in I_1$$

So from theorem 4.1 we get $\pi(k) = c$, where c is a constant.

Since $\{\pi(i)\}$ is a distribution on I_1 we obtain $c = \frac{b}{r}$ and from (10) $E(T_1) = \frac{1}{\pi(o)} = \frac{r}{b}$ and this completes the proof of the theorem 3.1 which establishes theorem 1.1.

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