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Conditionally Infinitesimal Systems of Random Variables

K. L. D. GUNAWARDENA*
Statistical Laboratory, Department of Mathematics, University of Manchester, England.

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Abstract : Infinitestimal systems of random variables, defined for a sequence of independent random variables are an important class of random variables in Probability Theory. In this paper, conditionally infinitesimal systems are defined for a sequence of dependent random variables and some properties are obtained.

1. Introduction

Let $\{\{X_{n,k}\}\}$ $k = 1, 2, \ldots\ldots, k_n$; $n = 1, 2, \ldots\ldots$ be a double sequence of random variables which are row-wise independent. That is for $n = 1, 2\ldots\ldots$; $k_n \to \infty$ as $n \to \infty$ and for every n (which denotes the row) the random variables $X_{n,1}, X_{n,2} \cdots X_{n,kn}$ are independent.

Definition 1.1

A sequence of independent random variables $\{\{X_{n,k}\}\}$ is said to be infinitesimal if for every sequence of integers k which satisfy $1 \leqslant k \leqslant k_n$ for all n we have, $X_{n,k} \to 0$ in probability as $n \to \infty$.

Notation.

$X_{n\,k} \to 0$ in probability as $n \to \infty$ will be written as $X_{n,k} \xrightarrow[n \to \infty]{P} 0$.

Consider a double sequence $\{\{X_{n,k}\}\}$ $k = 1, 2\ldots\ldots, k_n$; $n = 1, 2\ldots\ldots$ random variables. For every $k\,(1 \leqslant k \leqslant k_n)$ and $n = 1, 2, \ldots\ldots$ we have an increasing sequence of σ — fields $F_{n,0} \subset F_{n,1} \subset \ldots\ldots \subset F_{n,k_n}$ such that every $X_{n,k}$ is $F_{n,k}$ measurable.

We shall extend the definition 1.1 to a sequence of dependent random variables and define "conditionally infinitestimal" as follows.

Definition 1.2

A sequence of random variables $\{\{X_{n,k}\}\}$ is said to be conditionally infinitestimal if

$$\max_{1 \leqslant k \leqslant k_n} P(\,|X_{n,k}| > \epsilon \mid F_{n,k-1}) \xrightarrow[n \to \infty]{P} 0 \qquad (1)$$

for every $\epsilon > 0$.

*Presently at the Department of Mathematics, University of Peradeniya, Peradeniya.

If the $\{\{X_{n,k}\}\}$ are independent then it can be shown that the above definition is equivalent to definition 1.1 and in the general case implies it. [3]

2. Equivalent Definition and Properties.

Lemma 2.1

$\{\{X_{n,k}\}\}$ is conditionally infinitesimal if and only if

$$\max_{1 \leqslant k \leqslant k_n} E\left(\frac{X^2_{n,k}}{1+X^2_{n,k}} \mid F_{n,k-1}\right) \xrightarrow[n \to \infty]{P} 0 \tag{2}$$

Proof :

Let $F(x \mid F_{n,k-1})$ be a regular conditional distribution for $X_{n,k}$ given $F_{n,k-1}$. [1]

$$E\left(\frac{X^2_{n,k}}{1+X^2_{n,k}} \mid F_{n,k-1}\right) = \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} F(dx \mid F_{n,k-1})$$

$$\leqslant \int_{|x| \leqslant \epsilon} x^2 F(dx / F_{n,k-1}) + \int_{|x| > \epsilon} F(dx \mid F_{n,k-1})$$

$$\leqslant \epsilon^2 + P(|X_{n,k}| > \epsilon \mid F_{n,k-1}) \tag{3}$$

Taking $\max\limits_{1 \leqslant k \leqslant k_n}$ and first letting $n \to \infty$ and then $\epsilon \to 0$ in (3) we get (2) if (1) is true.

Also $E\left(\frac{X^2_{n,k}}{1+X^2_{n,k}} \mid F_{n,k-1}\right) \geqslant \int_{|x| > \epsilon} \frac{x^2}{1+x^2} F(dx \mid F_{n,k-1})$

$$\geqslant \frac{\epsilon^2}{1+\epsilon^2} \int_{|x| > \epsilon} F(dx \mid F_{n,k-1})$$

$$= \frac{\epsilon^2}{1+\epsilon^2} P(|X_{n,k}| > \epsilon \mid F_{n,k-1})$$

Hence

$$P(|X_{n,k}| > \epsilon \mid F_{n,k-1}) \leqslant \frac{1+\epsilon^2}{\epsilon^2} \cdot E\left(\frac{X^2_{n,k}}{1+X^2_{n,k}} \mid F_{n,k-1}\right)$$

which implies (1) if (2) is true.

Note : If the $\{\{X_{n,k}\}\}$ are independent then lemma 2.1 reduces to lemma 1,[2]

Theorem 2.2

If the double sequence $\{\{X_{n,k}\}\}$ of non-negative random variables is conditionally infinitestimal then for every $t > 0$.

$$\max_{1 \leqslant k \leqslant k_n} E(1 - e^{-tX_{n,k}} \mid F_{n,k-1}) \xrightarrow[n \to \infty]{P} 0$$

Proof :

$$E(1 - e^{-tX_{n,k}} \mid F_{n,k-1}) = \int_0^\infty (1 - e^{-tx}) F(dx \mid F_{n,k-1})$$

$$= \int_0^\epsilon (1 - e^{-tx}) F(dx \mid F_{n,k-1}) + \int_\epsilon^\infty (1 - e^{-2tx}) F(dx \mid F_{n,k-1})$$

$$\leqslant t\epsilon + P(\mid X_{n,k} \mid > \epsilon \mid F_{n,k-1}) \tag{4}$$

Theorem 2.2 is proved by taking $\max_{1 \leqslant k \leqslant k_n}$ and first letting $n \to \infty$ and then $\epsilon \to 0$ in (4).

Theorem 2.3

If the double sequence $\{\{X_{n,k}\}\}$ is conditionally infinitesimal then

$$\max_{1 \leqslant k \leqslant k_n} \mid E(e^{itX_{n,k}} - 1 \mid F_{n,k-1}) \xrightarrow[n \to \infty]{P} 0.$$

Proof :

$$\mid E(e^{itX_{n,k}} - 1 \mid F_{n,k-1}) \mid = \mid \int_{-\infty}^{\infty} (e^{itx} - 1) F(dx \mid F_{n,k-1}) \mid$$

$$\leqslant \int_{|x| \leqslant \epsilon} \mid e^{itx} - 1 \mid F(dx / F_{nk,-1}) + \int_{|x| > \epsilon} \mid e^{itx} - 1 \mid F(dx \mid F_{n,k-1})$$

$$\leqslant \epsilon \mid t \mid + 2.P(\mid X_{n,k} \mid > \epsilon \mid F_{n,k-1}) \tag{5}$$

The proof of the theorem is complete by taking $\max_{1 \leqslant k \leqslant k_n}$ first letting $n \to \infty$ and then $\varepsilon \to 0$ in (5)

Note : If the $\{\{X_{n,k}\}\}$ are independent theorem 2.3 reduces to Theorem 1.[2]

3. Generalization of results in Section 2 to d — dimension (d > 1).

Let $\{\{\underline{X}_{n,k}\}\}$ be a double array of R^d valued random vectors.

Definition 3.1

A sequence of random vectors $\{\{\underline{X}_{n,k}\}\}$ is said to be conditionally infinitesimal if and only if

$$\max_{1 \leq k \leq k_n} P(\|X_{n,k}\| > \epsilon \mid F_{n,k-1}) \xrightarrow[n \to \infty]{P} 0 \tag{6}$$

for every $\epsilon > 0$, where $\|x\| = (x_1^2 + x_2^2 + \ldots\ldots + x_d^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, \ldots\ldots, x_d)$ ε Rd.

Lemma 3.2

$\{\{X_{n,k}\}\}$ is conditionally infinitesimal if and only if

$$\max_{1 \leq k \leq k_n} E\left(\frac{\|\underline{X}_{n,k}\|^2}{1 + \|\underline{X}_{n,k}\|^2} \mid F_{n,k-1}\right) \xrightarrow[n \to \infty]{P} 0.$$

Theorem 3.3

If the double array $\{\{\underline{X}_{n,k}\}\} \epsilon$ R + $\underline{d}$ is conditionally infinitestimal then for every $t \epsilon$ R + $\underline{d}$

$$\max_{1 \leq k \leq k_n} E(1 - e^{-(\underline{t}.\underline{X}_{n,k})} \mid F_{n,k-1}) \xrightarrow[n \to \infty]{P} 0.$$

Lemma 3.2 and theorem 3.3 are multidimensional versions of lemma 2.1 and theorem 2.2 respectively and the proofs are similar to the proofs in one-dimensional case.

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