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The Set of Numbers { 1, 5, 10 }

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Abstract : The set of numbers { 1, 5, 10 } has the property that, the product of any two decreased by 1, is a square. It is shown that there exists no positive integer c such that the set { 1, 5, 10, c } possesses the same property.

1. Introduction

The set of numbers [1, 3, 8, 120] has the property that, the product of any two increased by 1, is a square. Baker and Davenport[1] have proved that the property does not hold, if 120 is replaced by any other positive integer. The proof is based on Baker's Theorem in Diophantine Approximation. Kanagasabapathy and Ponnudurai[2] have described another method using nothing deeper than quadratic reciprocity, by which the result may be obtained. In this paper, we deal with the numbers 1, 5 and 10 which have the property that the product of any two decreased by 1, is a square. We prove the following theorem concerning these numbers :

2. Statement of the Theorem

Theorem : There exists no positive integer c such that the product of every pair of numbers of the set [1, 5, 10, c] decreased by one, is a perfect square.

Proof :

To establish our result, it is sufficient to show that the equations

$$c - 1 = x^2$$

and

$$5c - 1 = y^2$$

$$10c - 1 = z^2$$

cannot hold simultaneously for any positive integral values of x, y, z and c.

These equations lead to

$$y^2 - 5x^2 = 4 \tag{1}$$

and

$$z^2 - 2y^2 = 1 \tag{2}$$

All the non-negative integral solutions of the Pell Equation (2) are obtained from the following formula :

$$z_n + \sqrt{2}y_n = (1 + \sqrt{2})^{2n} \tag{3}$$

where n is a positive integer or zero.

Using (3) we obtain easily the following relations :

$$z_{m+n} = z_m z_n + 2y_m y_n$$

$$y_{m+n} = y_m z_n + y_n z_m$$

$$z_{-n} = z_n, \; y_{-n} = -y_n$$

$$z_{2n} = z_n^2 + 2y_n^2 = 2z_n^2 - 1 = 1 + 4y_n^2$$

$$y_{2n} = 2y_n z_n.$$

The following congruence holds :

$$y_{n+2r} \equiv -y_n \pmod{z_r} \tag{4}$$

We need the following results which can be easily established by induction :

(i) z_n is odd.

(ii) If $k = 2^t$, where t is an integer, then

$$z_{4k} \equiv 1 \pmod 8, \text{ for } t \geqslant 0 \tag{5}$$

$$z_{4k} \equiv 2 \pmod 5, \text{ for } t \geqslant 0 \tag{6}$$

$$z_{4k} \equiv -1 \pmod{17}, \text{ for } t = 0 \tag{7}$$

$$z_{4k} \equiv 1 \pmod{17}, \text{ for } t \geqslant 1 \tag{8}$$

We require the following table of values :

TABLE 1

n	y_n	z_n
0	0	1
1	2	3
2	12	17
3	70	99

From (1) we obtain

$$(5x_n)^2 = 5(y_n^2 - 4). \tag{9}$$

The proof is now accomplished in three stages :

(*a*) (9) is impossible if $n \equiv 0 \pmod 4$,
For, using (4) we obtain

$$y_n \equiv 0 \pmod{17}$$

Thus, we find that

$$(5x_n)^2 \equiv -20 \pmod{17}$$

and since $(-20 \mid 17) = -1$, (9) is impossible.

(*b*) (9) is impossible if $n \equiv 2 \pmod 4$.
For, using (4) we obtain

$$y_n \equiv \pm 12 \pmod{17}$$

Thus, we find that

$$(5x_n)^2 \equiv 20 \pmod{17}$$

and since $(20 \mid 17) = -1$, (9) is impossible.

(*c*) (9) is impossible if $n \equiv \pm 1 \pmod 4$, $n \neq \pm 1$, that is $n = \pm 1 + 4k + 8rk$, where r is an integer and $k = 2^t$ with t an integer $\geqslant 0$.

For, using (4) we obtain

$$\begin{aligned} y_n &\equiv \pm y_{4k+1} \pmod{z_{4k}} \\ &\equiv \pm 3y_{4k} \pmod{z_{4k}} \end{aligned}$$

Which implies

$$\begin{aligned} 2y_n^2 &\equiv 9(z_{4k}^2 - 1) \\ &\equiv -9 \pmod{z_{4k}} \end{aligned}$$

Thus we find that

$$\begin{aligned} (10x_n)^2 &= 10(2y_n^2 - 8) \\ &\equiv 10(-17) \pmod{z_{4k}} \\ &\equiv -2.5.17 \pmod{z_{4k}} \end{aligned}$$

and since

$$\begin{aligned} \left(\frac{-2.5.17}{z_{4k}}\right) &= \left(\frac{-1}{z_{4k}}\right)\left(\frac{2}{z_{4k}}\right)\left(\frac{5}{z_{4k}}\right)\left(\frac{17}{z_{4k}}\right) \\ &= \left(\frac{2}{5}\right)\left(\frac{\pm 1}{17}\right) \quad \text{using the congruences (5) to (8)} \\ &= -1, \end{aligned}$$

and so (9) is impossible.

Hence there exist no integral values x, y, z which satisfy equations (1) and (2) simultaneously. The theorem now follows.

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References

1. BAKER, A. & DAVENPORT, H. (1969). "The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$", *Q. Jl. Math.* (Oxford) **2**: 20 : 129-37.

2. KANAGASABAPATHY, P. & PONNUDURAI THARMAMBIKAI (1975). 'The simultaneous Diophantine Equations $y^2 - 3x^2 = -3$ and $z^2 - 8x^2 = -7$' *Q. Jl. Math.* (Oxford) **2**: 26: 275-278.