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Absolute Convergence Factors for Cesàro Means

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Abstract : The problem considered in this paper is an extension of the problem (A) : What conditions on the convergence factor $g(x)$ are necessary and sufficient in order that the product $f(x)\,g(x)$ is absolutely Cesàro limitable of order r whenever $f(x)$ is absolutely Cesàro limitable of order k ? We find necessary and sufficient conditions in order that, for some l' ; $f(x)\,g(x) \sim l'\,x^{p+q}\mid C, r\mid$ whenever $f(x) \sim lx^{p} \mid C, k\mid$ for some l, where $p > -1$, $p+q > -1$, $r, k \in N$, $r \geqq k$, and f, g satisfy suitable local conditions. When $p = q = 0$, the conditions we obtain in this case are precisely those required in problem (A).

1. Introduction

We have considered[6] the problem of finding conditions which are necessary and sufficient in order that $f(x)\,g(x) = 0(x^{p+q})\mid C, r\mid$ whenever $f(x) = 0(x^{p})$ $\mid C, k\mid$, where $r, k \in Z^{+}$, $r \geqq k$, $p, q \in \mathbb{R}$, $p + q \leqq -1$, f is locally Lebesgue integrable and g^{k-1} is locally absolutely continuous. We showed that in this case, the convergence factor g belongs to a restricted class, viz., a subclass of the class of all polynomials.

In this paper, we consider the case $p + q > -1$ and the problem in a slightly more general form. We find conditions necessary and sufficient in order that for some l', $f(x)\,g(x) \sim l'\,x^{p+q}\mid C, r\mid$ whenever $f(x) \sim lx^{p}\mid C, k\mid$ for some l, where $p > -1$, $p + q > -1$, $r, k \in Z^{+}$, $r \geqq k$, g is bounded and measurable locally when $k = 0$, and g^{k-1} is locally absolutely continuous when $k \geqq 1$. The sequence analogue of this problem for $p = q = 0$ has been considered in (5).

The following are some preliminary definitions :

If $f \in L_{loc}$, $k \in \mathbb{N}$, define $I_k f(x) = f_k(x) = \displaystyle\int_1^x \frac{(x-t)^{k-1}}{(k-1)\,!}\, f(t)\,dt$, and $f_0(x) = f(x)$.

If $p > -1$, we write $f(x) \sim lx^{p}\mid C, k\mid$ if

$(p+1)\,(p+2)\,\ldots\,(p+k)\,x^{-p-k} f_k(x) \to l$ as $x \to \infty$ and $x^{-p-k} f_k(x) \in BV(1, \infty)$.

In particular, $f(x)$ is absolutely Cesàro limitable of order k to l if $k\,!\,x^{-k} f_k(x) \to l$ as $x \to \infty$ and $x^{-k} f_k(x) \in BV(1, \infty)$.

2. Statement of the Theorems

We assume throughout that $p > -1$, $p + q > -1$ and the local conditions stated above hold.

Theorem 1. *(a)* If $r = k = 0$, a necessary and sufficient condition that $f(x)\,g(x) \sim lx^{p+q}\ |C, 0|$ whenever $f(x) \sim lx^{p}\ |C, O|$ is that

$$x^{-q}\, g(x) \in BV(1, \infty).$$

(b) If $k = 0$, $r = 1$, a n.a.s.c. that $f(x)\,g(x) \sim l'\, x^{p+q}\ |C, 1|$ whenever $f(x) \sim lx^{p}\ |C, 0|$ is $(i)_b : x^{-p-q-1}\, Ix^{p}\, g(x) \in BV(1, \infty)$.

(c) If $k = 0$, $r > 1$, then conditions n.a.s. in order that $f(x)\,g(x) \sim l'\, x^{p+q}\ |C, r|$ whenever $f(x) \sim lx^{p}\ |C, 0|$ are :

(i)$_c$ $\displaystyle\int_1^x t^{-q}\, g(t)\, dt = 0(x)$ as $x \to \infty$,

(ii)$_c$ $x^{-p-q-r}\, I_r\, x^{p}\, g(x) \in BV(1, \infty)$.

Theorem 2. If $r \geq k \geqq 1$, conditions n.a.s. that $f(x)\,g(x) \sim l'\, x^{p+q}\ |C, r|$ whenever $f(x) \sim lx^{p}\ |C, k$ are :

(i) $g(x) = 0(x^{q})$ as $x \to \infty$

(ii) $g^{k-1}(x) = 0(x^{q+1-k})$ as $x \to \infty$

(iii) $x^{-p-q-r}\, I_r\, x^{p}\, g(x) \in BV(1, \infty)$.

3. Auxiliary Results

In this section we give some results which will be used in the proofs of the theorems.

Lemma 1 If $d_{ij} = \dfrac{\theta^{r+i-j}}{(r+i-j)!}$, $i, j = 1, 2, \dots r-1$, $r \in \mathbb{N}$, then

$D = det\,(d_{ij})_{(r-1)\times(r-1)} = K\,\theta^{r(r+1)}$ where $k \neq 0$, and D_{ij}, the cofactor of d_{ij} in D is given by $D_{ij} = K_{ij}\,\theta^{(r-2)\,r+j-i}$, where K_{ij} is independent of θ.

This result is easily proved, by induction with respect to r.

Lemma 2 If $n \in \mathbb{N}$, $\phi^{n} \in AC_{loc}$, $\phi(x) = 0(x^{q})$ as $x \to \infty$, and $\phi^{n}(x) = 0(x^{q-n})$ as $x \to \infty$, then $\phi^{j}(x) = 0(x^{q-j})$ for $j = 0, 1, \dots n$.

See (1), page 309 and (7), Lemma 1(c).

Lemma 3 Let $a(x,t) \in L(1,x)$ for $1 \leqq t \leqq x$, $a(x,t) = 0$ for $t > x$, and $v(x) = \int_1^x a(x,t)\, s(t)\, dt$. Then a n.a.s.c. that $v \in BV\,(1,\infty)$ whenever $s \in BV\,(1,\infty)$ is that there exists H independent of t such that

$$\int_t^\infty |\, d_x A(x,t)\,| \leqq H \text{ for all } t \geqq 1, \text{ where } A(x,t) = \int_t^x a(x,u)\, du.$$

This result follows from (4) Theorem 3 after an integration by parts. Cf. Also (3) — VIII. Note that $A(t,t) = A(t+,t) = 0$ in this case.

Lemma 4 Let $a(x,t)$, $A(x,t)$, $v(x)$ be as in Lemma 3. Let $n \in Z^+$ and s^n be the class of all real functions on $[1,\infty)$ such that $s^n(t)$ is absolutely continuous, locally and $s \in BV(1,\infty)$. Then, if $V(t) = \int_t^\infty |\, d_x A(x,t)\,| \in L\,(1,T)$ for every $T > 1$, a necessary and sufficient condition that $v \in BV(1,\infty)$ whenever $s \in S^n$ is that there exists H independent of t, and a constant T such that

$$V(t) \leqq H \text{ for almost all } t \geqq T.$$

This result follows from the theorem proved in (9) after an integration by parts. Cf. also (8), Theorem 4.

Lemma 5 If $p + k > -1$, $k' > k$ and if $g(x) \sim lx^p \mid C, k \mid$, then

$$g(x) \sim lx^p \mid C, k' \mid.$$

'$\sim lx^p$' may be replaced by '$=$ 'o' (x^p)' or '$= 0\,(x^p)$' here. Cf. (2), Lemma 3.

Lemma 6 If $p + k > -1$, $p + q > -1$ and $g(x) \sim lx^p \mid C, k \mid$, then

$$x^q g(x) \sim lx^{p+q} \mid C, k \mid.$$

'$\sim lx^p$' may be replaced by '$= o\,(x^p)$' or '$= 0\,(x^p)$' here. Cf. (2), Lemma 4.

4. Proofs of the Theorems

Theorem 1(*a*) is well known and its proof is omitted. We note at the outset that conditions $(i)_b$, $(ii)_c$ and (iii) are necessarily satisfied, since we may take $f(x) = lx^p$, $l \neq 0$, in particular. Hence we assume that $x^{-p-q-r} I_r\, x^p g(x) \in BV(1,\infty)$ in what follows. (1)

Case I Let $r > k$.

By repeated partial integration we have, when $k \geqq 1$,

$$I_r f(x)\, g(x) = \int_1^x \frac{(x-t)^{r-1}}{(r-1)!} f(t)\, g(t)\, dt = (-1)^k \int_1^x f_k(t)\, D^k_t\, G_r(x,t)\, dt \qquad (2)$$

where $D_t = \frac{\partial}{\partial t}$ and $G_r(x, t) = \frac{(x-t)^{r-1}}{(r-1)!} g(t)$.

This formula holds for $k = 0$ too.

Define $s(t) = t^{-p-k} f_k(t)$, $a(x, t) = (-1)^k t^{p+k} x^{-p-q-r} D^k_t G_r(x, t)$ and

$$A(x, t) = \int_t^x a(x, u)\, du.$$

Then, $$x^{-p-q-r} I_r f(x) g(x) = v(x) = \int_1^x a(x, t) s(t)\, dt, \tag{3}$$

and we want n.a.s.c. in order that $v \in BV(1, \infty)$ whenever $s \in BV(1, \infty)$ when $k = 0$, and whenever $s \in S^{k-1}$ when $k \geqq 1$.

Now, $A(x, t) = \int_1^x x^{-p-q-r} u^{p+k} D^k_u G_r(x, u)\, du - \int_1^t x^{-p-q-r} u^{p+k} D^k_u G^r(x, u)\, du$

$$= (-1)^k x^{-p-q-r} (p+1) \ldots (p+k) I_r x^p g(x) - x^{-p-q-r} \sum_{m=0}^{r-1} c_m x^{r-m-1} \int_1^t u^{p+k} D^k u^m g(u)\, du \tag{4}$$

where $c_m = \frac{(-1)^m}{(r-1)!} \binom{r-1}{m}$, by partial integration and the binomial theorem.

By (1) and (4) it follows that $V(t)$ is bounded and hence Lebesgue integrable locally.

Thus, applying Lemma 3 when $k = 0$ and Lemma 4 when $k \geqq 1$, we get the n.a.s.c. $\int_t^\infty | d_x A(x, t) | = 0(1)$, which, by (1) and (4) reduces to

$$\int_t^\infty | d_x (x^{-p-q-r} \int_1^t u^{p+k} D^k_u G_r(x, u)\, du) | = 0(1) \tag{5}$$

When $k = 0$, $r = 1$, it immediately follows that $(i)_b$ implies (5), and hence $(i)_b$ is both necessary and sufficient.

When $k = 0$, $r > 1$ repeated partial integration gives

$$\int_1^t \frac{(x-u)^{r-1}}{(r-1)!} u^p g(u)\, du = \sum_{m=1}^{r} \frac{(x-t)^{r-m}}{(r-m)!} I_m t^p g(t) \tag{6}$$

Since (1) holds, $\int_t^\infty | d_x (x^{-p-q-r} I_r t^p g(t)) | = | I_r t^p g(t) | t^{-p-q-r} = 0(1)$, and hence, by (6), condition (5) reduces to

$$\int_t^\infty | d_x \left(x^{-p-q-r} \sum_{m=1}^{r-1} \frac{(x-t)^{r-m}}{(r-m)!} I_m t^p g(t) \right) | = 0(1) \qquad (7)$$

If $(i)_c$ holds, then by Lemmas 4 and 5, we get

$$t^p g(t) = 0(t^{p+q})(C, m) \text{ for } m = 0, 1, \dots r-1.$$

Hence $\int_t^\infty | d_x (x^{-p-q-r} (x-t)^{r-m} I_m t^p g(t)) | = | I_m t^p g(t) | \int_t^\infty | d_x (x^{-p-q-r} (x-t)^{r-m} | = 0(t^{p+q+m}).\ A_1 t^{-p-q-m} = 0(1)$ for every m, showing that (7) holds.

Thus $(i)_c$ and $(ii)_c$ are sufficient in this case, and only the necessity of $(i)_c$ need be established.

By Lemma 5, since $f(x) g(x) \sim 1' x^{p+q} | C, r+1 |$ for i = 0, 1,..., (7) holds with r replaced by $r+i$.

Thus $$\int_t^\infty | d_x \left\{ x^{-p-q-r-i} \left(\sum_{j=1}^{r-1} + \sum_{j=r}^{r+i-1} \right) \left[\frac{(x-t)^{r+i-j}}{(r+i-j)!} I_j t^p g(t) \right] \right\} | = 0(1),$$

which reduces to

$$\int_t^\infty | d_x \left(x^{-p-q-r-i} \sum_{j=1}^{r-1} \frac{(x-t)^{r+i-j}}{(r+i-j)!} I_j t^p g(t) \right) | = 0(1), \qquad (8)$$

since $\int_t^\infty | d_x (x^{-p-q-r-i} (x-t)^{r+i-j} I_j t^p g(t)) | = 0(t^{p+q+j}).\ A_2 t^{-p-q-j}$

$$= 0(1) \text{ for } j = r, \dots\dots r+i-1.$$

(8) gives $$\int_t^\infty | d_x \phi_i (x, t) | = 0(1), \qquad (9)$$

where $x^{p+q+r+i} \phi_i (x, t) = \sum_{j=1}^{r-1} \frac{(x-t)^{r+i-j}}{(r+i-j)!} I_j t^p g(t),\ i = 1, \dots r-1.$

Solving this system of linear equations for $I_j t^p g(t)$, we get

$I_j t^p g(t) = D^{-1} \sum_{i=1}^{r-1} D_{ij} x^{p+q+r+i} \phi_i (x, t)$ for $j = 1, \dots r-1$, where by

Lemma 1, $D = K(x-t)^{r(r-1)}$, $K \neq 0$, $D_{ij} = K_{ij}(x-t)^{(r-2)r+j-i}$, K_{ij} being independent of x and t.

Hence $It^p g(t) = K^{-1}(x-t)^{-r(r-1)} \sum_{i=1}^{r-1} D_{i1}\, x^{p+q+r+1}\, \phi_i(x,t)$ giving

$$\frac{K(x-t)^{r(r-1)}\, It^p\, g(t)}{x^{rt-r+1+p+q}} = \sum_{i=1}^{r-1} K_{i1}\, \phi_i(x,t)\left[\frac{(x-t)}{x}\right]^{(r-1)^2-i} \tag{10}$$

Now (9) implies that $\int_t^{\infty} \left| d_x \left\{ \frac{(x-t)^s}{x^s} \phi_i(x,t) \right\} \right| = 0(1)$ for any $s \in \mathbb{N}$.

Putting $s = (r-1)^2 - i$, we get $\int_t^{\infty} \left| d_x \left\{ \left[\frac{(x-t)}{x}\right]^{(r-1)^2-i} \phi_i(x,t) \right\} \right| = 0(1)$,

and thus (10) gives $\int_t^{\infty} \left| d_x \left\{ \frac{(x-t)^{r(r-1)}}{x^{r^2-r+1+p+q}}\, It^p\, g(t) \right\} \right| = 0(1)$.

i.e. $|It^p g(t)| . A_3\, t^{-p-q-1} = 0(1)$, which gives $(i)_c$ by Lemma 6, and thus Theorem 1 is complete.

Proof of Theorem 2 *Sufficiency.* We see that (i) and (ii) imply

$$g^j(t) = 0(t^{p-j}).\ j = 0, 1, \ldots k-1, \text{ by Lemma 2.} \tag{11}$$

Now, (5) may be written

$$\int_t^{\infty} \left| d_x \left\{ x^{-p-q-r} \int_1^t u^{p+k} du \left[\sum_{m=0}^{r-1} a_m x^{-r-m-1} \sum_{n=0}^{k} b_{mn}\, u^{m-n}\, g^{kn}(u) \right] \right\} \right| = 0(1) \tag{12}$$

where $a_m = (-1)^m \binom{r-1}{m}$ and $b_{mn} = \binom{k}{n} m(m-1)\ldots(m-n+1)$.

Now $\int_t^{\infty} \left| d_x \left\{ x^{-p-q-m-1} \int_1^t u^{p+k}\, u^{mn}\, g^{k-n}(u)\, du \right\} \right|$

$= \int_t^{\infty} \left| d_x\, x^{-p-q-m-1} \right| \left| 0(u^{p+q+m}) \right| = 0(1)$ for $m = 0, 1 \ldots$ and $n = 1 \ldots k$ by (11) and

$$\int_t^{\infty} | d_x \{ x^{-pq-m--1} \int_1^t u^{p+k+m} g^k (u)\, du \} |$$

$$= \int_t^{\infty} | d_x \{ x^{-p-q-m-1} ([u^{p+k+m} g^{k-1} (u)]_1^t - \int_1^t (p + k + m) u^{p+k+m+1} g^{k-1} (u)\, du) \} |$$

$= 0(1)$ for $m = 0\ 1 \ldots$ by (ii).

Thus (12) holds proving the sufficiency of (i) (ii) and (iii).

Necessity We show that (5) implies (i) and (ii).

The proof is by induction. Assume that $(5)_{k = k_1}$ implies (i) and $(\text{ii})_{k = k_1}$.

Suppose $k = k_1 + 1$. Then by lemma 5 (iii) and $(5)_{k = k_1 + 1}$ are necessary, and by the inductive assumption (i) and (ii) $k = k_1$ are necessary.

Also, $\int_1^t u^{p+k_1} D_u^{k_1} G_r (x, u)\, du + \int_1^t \frac{u^{p+k_1+1}}{p + k_1 + 1} D_u^{k_1} G_r (x, u)\, du$

$= \frac{t^{p+k_1+1}}{p + k_1 + 1} D_t^{k_1} G_r (x, t) + \sum_{m=0}^{r-1} a_m x^{r-m-1}$, where a_m is constant, and hence,

since $(\text{ii})_{k = k_1}$ and $(\text{ii})_{k = k_1 + 1}$ hold, it is necessary that

$$\int_t^{\infty} | d_x \{ x^{-p-q-r} t^{p+k_1+1} D_t^k G_r (x, t) \} |$$

$$= \int_t^{\infty} | d_x \left\{ x^{-p-q-r} t^{p+k_1+1} \sum_{k=0}^{k_1} (-1)^j \binom{k_1}{j} \frac{(x-t)^{r-j-1}}{(r-j-1)!} g^{k_1-j} (t) \right\} | = 0(1) \quad (13)$$

Now $\int_t^{\infty} | d_x \{ x^{-p-q-r} t^{p+k_1+1} (x - t)^{r-j-1} g^{k_1-j} (t) \} |$

$= t^{p+k_1+1} .0 (t^{q+j-k_1}). A_4\, t^{-p-q-j-1} = 0\ (1)$ for $j = 1, 2, \ldots k_1$ by (i) and $(\text{ii})_{k_1}$ (14)

(13) and (14) give $\int_t^{\infty} | d_x \{ x^{-p-q-r} t^{p+k_1+1} (x - t)^{r-1} g^{k_1} (t) \} | = 0\ (1)$ and hence $g^{k_1} (t) = 0\ (t^{q-k_1})$, which is $(\text{ii})_{k=k_1+1}$, and by induction, we get the necessity.

Case II $r = k$. Since (i) and (ii) are independent of r and are necessary when $r > k$, by Lemma 5. (i) and (ii) are *a fortiori* also necessary when $r = k$. We thus have to consider only the sufficiency. In this case we have

$$x^{-p-q-k} I_k f(x) g(x) = \int_1^x a(x\ t)\, s(t)\, dt + (-1)^k x^{-q} s(x) g(x) \tag{15}$$

and $$\int_1^x x^{-p-q-k} u^{p+k} D^k_u G_k(x\ u)\, du = (-1)^k x^{-p-q-k} (p+1) \ldots (p+k) I_k x^p g(x) - x^{-q} g(x) \tag{16}$$

by partial integration, instead of (3) and (4) respectively.

Now define $B(x, t) = A(x, t) + (-1)^k x^{-q} g(x)$.

Then, for fixed x, $-B(x, t)$ is still an indefinite integral of $a(x, t)$, and since

$$A(x, x) = 0, \quad \text{we have } B(x, x) = (-1)^k x^{-q} g(x).$$

Hence partial integration of (15) gives $x^{-p-q-k} I_k f(x) g(x) = \int_1^x B(x, t)\, ds(t)$,

and from this point onwards the proof of the sufficiency is exactly as in case I.

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