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The Diophantine Equation $(y(y+1))^2 = 3x(x+1)$

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Abstract : It is shown by using the properties of quadratic reciprocity that the only solution in positive integers of the Diophantine equation

$$(y(y+1))^2 = 3x(x+1)$$

is $x = 3$, $y = 2$.

The Equations

On putting $2x + 1 = V$ and $2y + 1 = Y$, the equation of the title reduces to

$$(Y^2 - 1)^2 = 12(V^2 - 1),$$

which implies that 6 divides $Y^2 - 1$.

Now, if we put $Y^2 - 1 = 6U$, the last equation becomes

$$V^2 - 3U^2 = 1. \tag{1}$$

The general solution of the equation (1) is given by

$$U_n = \frac{\alpha^n - \beta^n}{2\sqrt{3}}, \quad V_n = \frac{\alpha^n + \beta^n}{2} \tag{2}$$

where

$$\alpha = 2 + \sqrt{3}, \quad \beta = 2 - \sqrt{3}.$$

Hence, we must have

$$Y^2 = 6U_n + 1. \tag{3}$$

We obtain easily from (2) the following relations:

$$U_{m+n} = V_m U_n + V_n U_m, \tag{4}$$
$$V_{m+n} = V_m V_n + 3U_m U_n, \tag{5}$$
$$U_{-n} = -U_n, \quad V_{-n} = V_n, \tag{6}$$
$$U_{2n} = 2U_n V_n, \tag{7}$$
$$V_{2n} = V^2_n + 3U^2_n = 1 + 6U^2_n = 2V^2_n - 1, \tag{8}$$
$$U_{3n} = U_n(4V^2_n - 1), \tag{9}$$
$$V_{3n} = V_n(4V_n^2 - 3), \tag{10}$$
$$U_{5n} = U_n(16V_n^4 - 12V^2_n + 1), \tag{11}$$
$$V_{5n} = V_n f(V_n) = V_n h(U_n), \tag{12}$$

where $\quad f(V) = 16V^4 - 20V^2 + 5$

and $\quad h(U) = 144U^4 + 36U^2 + 1.$

The following congruences hold:

$$U_{n+2r} \equiv -U_n \pmod{V_r}, \tag{13}$$

$$U_{n+2r} \equiv U_n \pmod{U_r} \tag{14}$$

We need the following results which can be easily established by induction :

(i) U_n is odd or even according as n is odd or even ;

(ii) $U_2 \equiv 4 \pmod{13}$, and if $k = 2^t$ with t an integer > 1, $U_k \equiv \pm 4 \pmod{13}$, according as t is even or odd ; (15)

(iii) If $k = 2^t$, then $V_k \equiv 2, 7, 5, 3, -6 \pmod{23}$ when $t \equiv 0, 1, 2, 3, 4 \pmod{5}$ respectively. (16)

We also note that since $Y = 2y + 1$, Y is odd and so if U_n satisfies (3), then U_n must be even.

Hence, n must be necessarily even for (3) to hold.

We require the following table of values:

n	U_n	V_n
0	0	1
1	1	2
2	4	7
3	15	26
4	56	97
5	209	362
6	780	1351
7	2911	5042
8	10864	18817

The proof is now accomplished in five stages:

(*a*) (3) is impossible if $n \equiv 4 \pmod{10}$.

For, using (14) we find that

$$\begin{aligned} U_n &\equiv U_4 \pmod{U_5} \\ &\equiv 56 \pmod{209} \\ &\equiv 56 \pmod{11}, \text{ since } 11/209 \\ &\equiv 1 \pmod{11}. \end{aligned}$$

Thus we find that

$$6U_n + 1 \equiv 7 \pmod{11}$$

and since $(7/11) = -1$, (3) is impossible.

(*b*) (3) is impossible if $n \equiv 6 \pmod{10}$.

For, using (14) we find that

$$\begin{aligned} U_n &\equiv U_6 \pmod{U_5} \\ &\equiv 780 \pmod{209} \\ &\equiv 780 \pmod{11} \\ &\equiv -1 \pmod{11}. \end{aligned}$$

Thus we find that

$$6U_n + 1 \equiv -5 \pmod{11}$$

and since $(-5/11) = -1$, (3) is impossible.

(*c*) (3) is impossible if $n \equiv 8 \pmod{10}$.

For, using (14) we find that

$$\begin{aligned} U_n &\equiv U_8 \pmod{U_5} \\ &\equiv 10864 \pmod{209} \\ &\equiv 10864 \pmod{11} \\ &\equiv 7 \pmod{11}. \end{aligned}$$

Thus we find that:

$$\begin{aligned} 6U_n + 1 &\equiv 43 \pmod{11} \\ &\equiv -1 \pmod{11} \end{aligned}$$

and since $(-1/11) = -1$, (3) is impossible.

(*d*) (3) is impossible if $n \equiv 0 \pmod{10}$, $n \not\equiv 0$, that is, if $n = 5k(2m + 1)$ where $k = 2^t$ with t an integer $\geqslant 1$ and m is an integer.

For, using (13) we find that

$$\begin{aligned} U_n &\equiv \pm U_{5k} \pmod{V_{5k}} \\ &\equiv \pm U_k\,(16V^4_k - 12V^2_k + 1) \pmod{f(V_k)} \\ &\equiv \pm 4U_k\,(2V^2_k - 1) \pmod{f(V_k)} \end{aligned}$$

Whence, $6U_n + 1 \equiv \pm 24U_k\,(2V^2_k - 1) + 1 \pmod{f(V_k)}$

Now,

$$\left(\frac{1 \pm 24U_k(2V^2_k - 1)}{f(V_k)}\right) = \left(\frac{16V^4_k - 20V^2_k + 6 \pm 24U_k(2V^2_k - 1)}{f(V_k)}\right)$$

$$= \left(\frac{2}{f(V_k)}\right)\left(\frac{2V^2_k - 1}{f(V_k)}\right)\left(\frac{4V^2_k - 3 \pm 12U_k}{f(V_k)}\right)$$

$$= \left(\frac{12U^2_k + 1 \pm 12U_k}{h(U_k)}\right)$$

$$= \left(\frac{h(U_k)}{12U^2_k \pm 12U_k + 1}\right)$$

$$= \left(\frac{12 \cdot 13\, U^2_k}{12U^2_k \pm 12U_k + 1}\right)$$

$$= \left(\frac{13}{12U^2_k \pm 12U_k + 1}\right)$$

$$= \left(\frac{12U^2_k \pm 12U_k + 1}{13}\right)$$

$$= \left(\frac{-U^2_k \mp U_k + 1}{13}\right)$$

$$= \left(\frac{-16 \mp 4 + 1}{13}\right) \text{ by (15)}$$

$$= \left(\frac{2}{13}\right) = -1$$

and so (3) is impossible.

(*e*) (3) is impossible if $n \equiv 2 \pmod{10}$, $n \neq 2$, that is, $n = 2 + 10km$, where $k = 2^t$ with t an integer $\geqslant 0$ and m is an odd integer.

For, using (13) we find that

$$U_n = -U_2 \equiv -4 \pmod{V_{5k}}.$$

Thus,

$$6U_n + 1 \equiv -23 \pmod{V_k f(V_k)},$$

and so (3) would imply

$$\left(\frac{-23}{V_k}\right) = \left(\frac{-23}{f(V_k)}\right) = +1 \tag{17}$$

But it is easily seen that

$$V_2 \equiv 3 \pmod{4},$$
$$V_k \equiv 1 \pmod{4} \text{ for } t \geqslant 2,$$

and $f(V_k) \equiv 1 \pmod{4}$ for $t \geqslant 0$,

so, using quadratic reciprocity, (17) implies

$$\left(\frac{V_k}{23}\right) = +1 \text{ for } t \geqslant 1 \tag{18}$$

and $$\left(\frac{f(V_k)}{23}\right) = +1 \text{ for } t \geqslant 0. \tag{19}$$

Now by (16) the residues of V_k modulo 23 and so $f(V_k)$ modulo 23 are periodic with respect to t, and the length of the period is 5. The following table gives these residues and the signs of their Legendre symbols $(V_k/23)$ and $(f(V_k)/23)$; we see that (18) or (19) is impossible.

t	0	1	2	3	4	5	6
$V_k \bmod 23$		7	5	3	17	2	7
$f(V_k) \bmod 23$	20	20	6	17	11	20	20
$(V_k/23)$	+	–	–	+	–	+	–
$(f(V_k)/23)$	–	–	+	–	–	–	–

We have now two further cases $n = 0, 2$ to consider.
When $n = 0$, we find $x = 0$, $y = 0$, a trivial solution.
When $n = 2$, $U_2 = 4$, $V_2 = 7$.
Thus we get $x = 3$, $y = 2$.

Hence, $x = 3$, $y = 2$ is the only solution of the given equation in positive integers. Now we can write down the complete solution in integers; it consists of the four trivial pairs of solutions obtained by equating both sides of the equation of the title to zero and the four pairs given by the following table :

x	y
3	2
3	–3
–4	2
–4	–3

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References

1. T. Nagell, Introduction to Number Theory, Wiley, New York, 1951.