J. Natn. Sci. Coun. Sri Lanka 1976 4(2) : 149—155

Absolute convergence factors for integrals

C. YOGACHANDRAN
Department of Mathematics, University of Sri Lanka,
Peradeniya Campus, Peradeniya, Sri Lanka

(*Paper accepted : 11th October 1976.*)

Abstract: The absolute convergence factor problem for integrals of finding conditions necessary and sufficient that $f(x)g(x) = 0(x^{p+q}) / C, r /$ whenever $f(x) = 0(x^p) / C, k /$ where r, k are non-negative integers such that $r \geqslant k$, p, q are real and $p + q \leqslant -1$, is considered here. It is shown that the conditions on the convergence factor g restrict it to being either zero or a polynomial.

1. Introduction

Tyler (1) has considered the following absolute convergence factor problem for sequences : If r, k are non-negative integers, what conditions are necessary and sufficient that the sequence $\{ \epsilon_n \}$ be such that $s_n \epsilon_n$ be limitable $| C, r |$ whenever s_n is limitable $| C, k |$? The method used by Tyler can be adapted to find the conditions on the sequence $\{ \epsilon_n \}$ in the more general problem : What conditions are necessary and sufficient that $s_n \epsilon_n = 0 (n^{p+q}) | C, r |$ whenever $s_n = 0 (n^p) | C, k |$, where p is real, $p + q > -1$? Tyler's method cannot be used to discuss the case $p + q \leqslant -1$, and this problem has not been considered before.

The author has considered the integral analogue of this problem in the more general case when p is real and q is real ; viz: What conditions are necessary and sufficient that $f(x) g(x) = 0 (x^{p+q}) | C, r |$ whenever $f(x) = 0 (x^p) | C, k |$ where p, q are real, $r \geqslant k$? The conditions on the convergence factor g in the case $p + q \leqslant -1$ turn out to be of character quite different to those in the case $p + q > -1$. It is shown that in the case $p + q \leqslant -1$, $f(x) = 0 (x^p) | C, k |$ can imply $f(x) g(x) = 0 (x^{p+q}) | C, r |$ only when g is zero or belongs to a restricted class of functions. The conditions in the case $p + q > -1$ do not restrict g to such a narrow class.

Leaving out the trivial case $r = k = 0$, it can be shown that if $p + q > -1$, the conditions on g are the following :

(I) If $k = 0$, $r \geqslant 1$, the only condition is that $g \in G_1$; i.e.

$$\int_1^x t^{-q} g(t) \, dt = 0 (x) \text{ as } x \to \infty.$$

(II) If $r \geqslant k \geqslant 1$, then $g \in G_2$; i.e.

(i) $g(x) = 0(x^q)$ as $x \to \infty$.

(ii) $g^{k-1}(x) = 0(x^{q+1-k})$ as $x \to \infty$.

The conditions in I and II turn out to be necessary even in the case $p + q \leqslant -1$, but a stronger result is proved here. It is shown that it is necessary that g belongs to classes smaller than G_1 and G_2 in the cases $k = 0$ and $k \geqslant 1$ respectively.

This paper deals with the case $p + q \leqslant -1$ in detail. The case $p + q > -1$ will be dealt with in a subsequent paper.

2. Preliminary Definitions

All functions considered here are real and defined on $[1, \infty)$. If f is Lebesgue integrable locally (i.e. in every finite subinterval of $[1, \infty)$), write $f \in L$. In what follows, take $f \in L$, and if $k = 0$, take g to be bounded and measurable locally, and if $k \geqslant 1$, take g^{k-1} to be absolutely continuous locally, g^m denoting m − th derivative of g, g^0 denoting g.

Let r, k be non-negative integers such that $k \geqslant 0$, $r \geqslant k$.

Let $f_m(x)$ be the m−th integral of $f(x)$; i.e. $I_m f(x) = f_m(x) = \int_1^x f_{m-1}(t)\,dt$ for $m \geqslant 1$, $I_0 f(x) = f_0(x) = f(x)$.

Then, $f_m(x) = \int_1^x \frac{(x-t)^{m-1}}{(m-1)!} f(t)\,dt$ for $m \geqslant 1$.

If p is real, write $f(x) = 0(x^p) \mid C, k \mid$ if $x^{-p-k} f_k(x) \in BV[1, \infty)$ and is $0(1)$ as $x \to \infty$. In this paper, take p to be real, $p + q \leqslant -1$.

Theorem. (i) If $k = 0$, $r \geqslant 1$, a necessary and sufficient condition that $f(x)\,g(x) = 0(x^{p+q}) \mid C, r \mid$ whenever $f(x) = 0(x^p) \mid C, k \mid$ is that

(a) $g(x) = 0$ for almost all $x \geqslant 1$.

(ii) If $r \geqslant k \geqslant 1$, let M be the greatest integer such that $M < p + q + k + 1$; $M \leqslant q$, except in the case $r = k$, $p + q + k + 1 \leqslant 0$, $q \geqslant 0$, when M is taken to be zero. If M turns out to be negative, this is taken to mean $g(x) = 0$. Then, a necessary and sufficient condition that $f(x)\,g(x) = 0(x^{p+q}) \mid C, r \mid$ whenever $f(x) = 0(x^p) \mid C, k \mid$ is that (*b*) $g(x)$ is a polynomial in x of degree not exceeding M.

Note. If $M \geqslant 1$, then (ii) means that $g(x)$ is a polynomial of degree less than or equal to M, and if $M \leqslant 0$, then $g(x)$ is constant.

3. Subsidiary Results

Lemma 1. Let N be a positive integer, $X > 1$, $\theta \in L[1, X]$. Then, a necessary condition that $\int_1^x \theta(t)\, s(t)\, dt = 0$ whenever $s^N \in L$ and $s(X) = s^1(X) = \ldots = s^{N-1}(X) = 0$ is that $\theta(t) = 0$ p.p in $[1, X]$.

Proof. Assume that the conclusion is false. Then,

$$\int_1^x |\theta(t)|\, dt = 5K, \text{ where } K > 0. \tag{1}$$

There exists a step function $u(t)$ with a finite number of steps in $[1, X]$ such that $\int_1^x |\theta(t) - u(t)|\, dt < K$. (2)

(1) and (2) give $\int_1^x |u(t)|\, dt > 4K$. (3)

Since $u \in L[1, X]$ there exists $m^1 > 0$ such that, for any measurable subset e of $[1, X]$ with $m\, e < m^1$, we have $\int_e |u(t)|\, dt < K$. (4)

Clearly, it is possible to select a finite set of pairwise disjoint intervals in $[1, X]$ not containing any discontinuity of $u(t)$, such that their union H satisfies $m\{[1, X] - H\} < m^1$.

Thus (4) gives $\int_1^x |u(t)|\, dt - \int_H |u(t)|\, dt < K$ (5)

Now, there exists s such that $s^N \in L$, $s(X) = s^1(X) = \ldots s^{N-1}(X) = 0$ and $s(t) = \operatorname{sgn} u(t)$ in H, (6)

$|s(t)| \leqslant 1$ in $[1, X] - H$. (7)

From (7), (2), (6), (5) and (3) used in that order, it follows that

$\int_1^X \theta(t)\, s(t)\, dt > K$, which is a contradiction, and hence the lemma.

Lemma 2. A necessary condition that $\int_1^X \theta(t)\, s(t)\, dt = 0$, where $X > 1$, $\theta \in L(1, X)$ whenever $s \in L$ and $s(X) = 0$ is that $\theta(t) = 0$ p.p in $[1, X]$.

Proof. Proceedings as in Lemma 1, we see that there exists a step function

$u(t)$ such that $\int_1^X |u(t)|\, dt > 4K$.

Clearly, there exists s such that $s \in L$, $s(X) = 0$ and $s(t) = \operatorname{sgn} u(t)$ in $[1, X)$.

Then, $\int_1^X \theta(t)\, s(t)\, dt > 3K > 0$, and hence the lemma.

Lemma 3. If $\phi \in BV[1, \infty)$ and $\delta > 0$, then $t^{-\delta} \int_1^t u^{\delta-1} \phi(u)\, du \in BV [1, \infty)$. See (2), Lemma 3 .

4. Proof of the Theorem

When $r \geqslant k \geqslant 1$, repeated partial integration gives

$$I_r f(x)\, g(x) := \int_1^x \frac{(x-t)^{r-1}}{(r-1)!} f(t)\, g(t)\, dt$$

$$= (-1)^k \int_1^x f_k(t)\, (D_t)^k\, G_r(x, t)\, dt + \delta_{rk} f_k(x)\, g(x), \tag{8}$$

where $D_t = \frac{\partial}{\partial t}$, $G_r(x,t) = \frac{(x-t)^{r-1}}{(r-1)!}\, g(t)$ and δ_{rk} is the Krönecker delta.

Formula (8) also holds when $k = 0$, $r \geqslant 1$.

We want conditions necessary and sufficient that $x^{-p-q-r} I_r f(x)\, g(x) \in BV[1, \infty)$ and is $0(1)$ as $x \to \infty$ whenever $x^{-p-k} f_k(x) \in BV\ [1, \infty)$ and is $0(1)$ as $x \to \infty$. (9)

Necessity. By (9), it is necessary that $x^{-p-q-r} I_r f(x)\, g(x) = 0\,(1)$ whenever $f \in L$ and $f_k(x) = 0$ for all $x \geqslant X$, where X is fixed, arbitrary and greater than 1, $k \geqslant 0$. (10)

For such f, $f_1(x) = f_2(x) = \ldots = f_{k-1}(x) = 0$ for all $x \geqslant X$ when $k \geqslant 1$,

and hence (8) gives $I_r f(x)\, g(x) = (-1)^k \int_1^x f_k(t)\,(D_t)^k G_r(x, t)\, dt$

for all $x \geqslant X$. (11)

$$\text{Hence } I_r fg = \sum_{m=1}^{r} c_m\, x^{r-m} \int_1^x f_k(t)\,(d/dt)^k\,[\,t^{m-1} g(t)\,]\, dt, \tag{12}$$

where $c_m = \dfrac{(-1)^{k+m-1}}{(r-1)!} \dbinom{r-1}{m-1}$.

Formula (12) holds for $k = 0$ too.

Thus (10) and (12) give : $\displaystyle\sum_{m=1}^{r} c_m\, x^{-p-q-m} \int_1^x f_k(t)\, D^k\,[t^{m-1} g(t)\,]\, dt = 0\,(1)$

whenever $f \in L$ and $f_k(x) = 0$ for all $x \geqslant X$. (13)

Let n be the integer such that $-n-1 < p+q \leqslant -n$ if $p+q > -r$, and $n = r$ if $p+q \leqslant -r$.

Then, $x^{-p-q-m} \to +\infty$ as $x \to \infty$ for $m = 1, 2, \ldots n$, and hence each coefficient of x^{-p-q-m} should be zero for $m = 1, 2, \ldots n$ in the series in (13).

Thus $\int_1^x f_k(t)\, D^k\,[\,t^{m-1} g(t)\,]\, dt = 0$ whenever $f \in L$ and $f_k(x) = 0$ for

all $x \geqslant X$, $m = 1, 2, \ldots n$. (14)

Write $N = k$, $s(t) = f_k(t)$, $\theta(t) = D^k\,[\,t^{m-1} g(t)\,]$ when $k \geqslant 1$, and $s(t) = f(t)$, $\theta(t) = t^{m-1} g(t)$ when $k = 0$.

Then by (14), $\int_1^x \theta(t)\, s(t)\, dt = 0$ whenever $s^N \in L$, $s(X) = s^1(X) = \ldots = s^{N-1}(X) = 0$ when $k \geqslant 1$, and $\int_1^x \theta(t)\, s(t)\, dt = 0$ whenever $s \in L$, $s(X) = 0$ when $k = 0$.

By lemmas 1 and 2 it follows that $\theta(t) = 0$ p.p in $[1, X]$. (15)

If $k = 0$, this gives $g(t) = 0$ p.p in $[1, X]$, and since X is arbitrary, it follows that $g(t) = 0$ for almost all $t \geqslant 1$, which is (a).

If $k \geqslant 1$, by (15), $D^k[t^{m-1} g(t)] = 0$ p.p. in $[1, X]$, for $m = 1.2 \ldots n$, and hence $g(t), tg(t), \ldots t^{n-1} g(t)$ are polynomials in t of degree at most $k - 1$. (16)

Thus $g(t)$ is a polynomial of degree at most $k - n$. (17)

If $n = r$, $r \geqslant k + 1$, (16) implies that $g(t) = 0$. (18)

If $n = r = k$, (16) implies that $g(t) =$ a constant. (19)

If $n < r$, then $-n-1 < p + q$ and hence $k - n < p + q + k + 1$.

Thus $g(t)$ is a polynomial of degree less than $p + q + k + 1$ if $p + q + k + 1 \geqslant 1$, by (17), and since it is also necessary that $g(t) = 0(t^q)$, (since $g \in G_2$), the required conclusion follows in the case $p + q + k + 1 > 0$.

If $r = k$, $p + q + k + 1 \leqslant 0$, $q \geqslant 0$, then $p + q + r = p + q + k < 0$ and hence $n = r$. Thus (19) implies that $g(t) =$ constant.

If $r = k$, $p + q + k + 1 \leqslant 0$ $q < 0$, again $g(t) =$ constant A, but $f(x) = 0(x^p)\ |C, k|$ can imply $Af(x) = 0(x^{p+q})\ |C, k|$ only if $A = 0$, i.e. $g(t) = 0$.

If $p + q + k + 1 \leqslant 0$, $r \geqslant k + 1$, by (18) it follows that $g(t) = 0$, which is the required conclusion, since $M < 0$ in this case.

This completes the necessity in (ii).

Sufficiency The sufficiency in (i) is trivial.

(ii) Omit the trivial case when $g(t) = 0$, and consider the case when $g(t)$ is a polynomial in t of degree at most M, where $M \leqslant q$, $M < p + q + k + 1$, $M \geqslant 0$.

Let $P^{(m)}(x)$ denote a polynomial in x of degree at most m, possibly different at each occurrence.

Then, $g(t) = P^{(M)}(t)$, and substituting for $g(t)$ and regrouping terms, we get $(D_t)^k G_r(x, t) = \sum_{i=0}^{M+r-1-k} t^i P^{(M+r-1-k-i)}(x).$

Hence (8) gives $I_r f(x) g(x) =$

$$\sum_{i=0}^{M+r-1-k} P^{(M+r-1-k-1)}(x) \int_1^x t^i f_k(t)\, dt + \delta_{rk} f_k(x) P^{(M)}(x). \qquad (20)$$

$$= \sum_{i=0}^{M+r-1-k} P^{(M+r-1-k-i)}(x)\,[0(x^{p+k+i+1}) + 0(\log x)] + 0(x^{p+k+M}).$$

$= 0(x^{p+M+r}) + 0(x^{M+r-1-k}\log x) + 0(x^{p+k+M}) = 0(x^{p+q+r})$, since $M + r - 1 - k < p + q + r$, $M \leqslant q$, $k \leqslant r$, except in the case $r = k$, $M = 0$, in which case $I_k f(x) g(x) = A f_k(x) = 0(x^{p+k}) = 0(x^{p+q+k})$. (21)

Now, since $M \leqslant q$, $t^{-p-k} f_k(t) \in BV[1, \infty)$, it follows that $t^{M-p-q-k} f_k(t) \in BV[1, \infty)$, and taking $\delta = p + q + k + 1 - M + i > 0$, $\phi(t) = t^{M-p-q-k} f_k(t)$ in Lemma 3, we see that $x^{M-p-q-k-1-i} \int_1^x t^i f_k(t)\, dt \in BV[1, \infty)$.

Hence from (20), it follows that $x^{-p-q-r} I_r f(x) g(x) \in BV[1, \infty)$.

This completes the sufficiency.

References

1. BORWEIN, D. (1950). *J. Lond. Math. Soc.* (25): 289-302.

2. TYLER, Mrs. B. (1958), *J. Lond. Math. Soc.* (33): 431-351.