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The Diophantine Equation $Y(Y+m)(Y+2m)(Y+3m)=3X(X+m)(X+2m)(X+3m)$

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Abstract : In the paper entitled "The Diophantine Equation Y (Y + 1) (Y + 2) (Y + 3) = 3X (X + 1) (X + 2) (X + 3)", published in the May 1975 issue of the Journal of the London Mathematical Society, it was shown that all the non-trivial solutions of the equation Y (Y + 1) (Y + 2) (Y + 3) = 3X (X + 1) (X + 2) (X + 3) are given by the following table :

X	2	2	5	5	—5	—5	—8	—8
Y	3	—6	7	—10	3	—6	7	—10

It is obvious that *m* times the above solution are solutions of the equation Y (Y + m) (Y + 2m) (Y + 3m) = 3X (X + m) (X + 2m) (X + 3m). The object of this paper is to provide conditions of a simple type on *m* under which the latter equation has no other non-trivial solution when *m* is a positive integer.

1. Introduction

It has been shown[2] that all thee non-trivial solutions of the equation

$$Y(Y+1)(Y+2)(Y+3)=3X(X+1)(X+2)(X+3) \qquad (1)$$

are given by the following table:

X	2	2	5	5	—5	—5	—8	—8
Y	3	—6	7	—10	3	—6	7	—10

It is obvious that m times the above solutions are solutions of the equation

$$Y(Y+m)(Y+2m)(Y+3m)=3X(X+m)(X+2m)(X+3m). \qquad (2)$$

The object of this paper is to provide conditions of a simple type under which the equation (2) has no other non-trivial solution when m is a positive integer.

2. The Equations

Substituting $x = 2X + 3m$, $y = 2Y + 3m$ in (2), we get

$$\left(\frac{y^2 - 5m^2}{4}\right)^2 - 3\left(\frac{x^2 - 5m^2}{4}\right)^2 = -2m^4.$$

Now, putting $V = \dfrac{y^2 - 5m^2}{4}$ and $U = \dfrac{x^2 - 5m^2}{4}$ we obtain the following equations :

$$V^2 - 3U^2 = -2m^4, \tag{3}$$

$$5m^2 + 4V = y^2, \tag{4}$$

and

$$5m^2 + 4U = x^2, \tag{5}$$

Hence, the equation (2) is equivalent to the equations (3), (4) and (5). It can be easily shown that if $(X, Y, m) = 1$, then $(U, V) = 1$ and conversely.

We shall call a solution X, Y of the equation (2) primitive if $(X, Y, m) = 1$.

We require the following lemmas :

Lemma 1. The equation (2) has no primitive solution if m is even.

Proof : Suppose that m is even and that the equation has a primitive solution.

Then (3) gives $V^2 - 3U^2 \equiv 0 \pmod 8$. Now, since $(U, V) = 1$, it follows from (3) that both U and V are odd.

Hence, $U^2 \equiv 1 \pmod 8$ and $V^2 \equiv 1 \pmod 8$, from which we have $V^2 - 3U^2 \equiv -2 \pmod 8$; a contradiction.

The lemma now follows.

Lemma 2. The equation (2) has no primitive solution if m has any prime factor $p \equiv 3, 5, 7 \pmod{12}$.

Proof. Suppose that the equation (2) has a primitive solution.

Then $(U, V) = 1$.

(i) Let $p = 3$.

Then by (3), $3 / V^2$ and therefore $3 / U$. Hence $(U, V) > 1$; a contradiction.

(ii) Let $p \equiv 5, 7 \pmod{12}$.

Then the Jacobi-Legendre symbol $(3/p) = -1$. Now since $(U, V) = 1$, from (3) it follows that $p \not| V$ and $p \not| U$. Again from (3), we obtain $V^2 \equiv 3U^2 \pmod p$, which implies that $(3/p) = 1$; a contradiction.

The lemma now follows.

Lemma 3. The equation (2) has no primitive solution if m has any prime factor $p \equiv 1 \pmod{12}$ such that

$$3^{(p-1)/4} \equiv -1 \pmod p.$$

Proof. Suppose the equation (2) has a primitive solution. Then $(U, V) = 1$ Now from (3), we have.

$$V^2 \equiv 3U^2 \pmod{p}. \tag{6}$$

From (4) and (5), V and U are quadratic residues of p and therefore by Euler's Criterion,

$$V^{(p-1)/2} \equiv U^{(p-1)/2} \equiv 1 \pmod{p} \tag{7}$$

Since $(U, V) = 1$, from (3) it follows that $p \nmid U$ and $p \nmid V$. By (6), we have

$$V^{(p-1)/2} \equiv 3^{(p-1)/4} . U^{(p-1)/2} \pmod{p}$$

and using (7), we have $3^{(p-1)/4} \equiv 1 \pmod{p}$; a contradiction. The lemma now follows.

Lemma 4. Every solution of (2), which is not a primitive solution, is a multiple of a primitive solution with a smaller m and conversely.

Proof. Suppose that X, Y, m satisfy (2) and that $(X, Y, m) = k > 1$. Dividing both sides of (2) by k^4, we have

$$\frac{Y}{k}\left(\frac{Y}{k}+\frac{m}{k}\right)\left(\frac{Y}{k}+2\frac{m}{k}\right)\left(\frac{Y}{k}+3\frac{m}{k}\right) = 3\frac{X}{k}\left(\frac{X}{k}+\frac{m}{k}\right)\left(\frac{X}{k}+2\frac{m}{k}\right)\left(\frac{X}{k}+3\frac{m}{k}\right)$$

and the lemma follows.

Corollary. If m is a prime, the non-primitive solutions of (2) are the solutions of the equations (1) multiplied by m. Now from lemmas (2), (3) and (4), we have the following theorem.

Theorem. The equation (2) has only the eight pairs of non-trivial solutions given by the following table :

X	$2m$	$2m$	$5m$	$5m$	$-5m$	$-5m$	$-8m$	$-8m$
Y	$3m$	$-6m$	$7m$	$-10m$	$3m$	$-6m$	$7m$	$-10m$

when m is an integer of the form $2^l . 3^r \prod_i p_i^{s_i} \prod_j q_j^{t_j}$, where l, r, s_i, t_j are non-negative integers and p_i's are positive primes $\equiv 5, 7 \pmod{12}$ and q_j's are positive primes $\equiv 1 \pmod{12}$ such that $3^{(q_j - 1)/4} \equiv -1 \pmod{q_j}$.

3. Discussion

Our theorem shows that for $m < 47$, no primitive solution exists except possibly for $m = 11$, 13 and 23. Now we shall discuss these three cases. When $m = 11$, the equation (2) has a primitive solution $X = -12$, $Y = -15$ and when $m = 13$, it has a primitive solution $X = -11$, $Y = -4$. When $m = 23$, the equation has no primitive solution, which can be proved as follows :

The equation (3), in this case reads as $V^2 - 3U^2 = -2.23^4$ whose complete solution (2) is given by :

$$V_n + U_n\sqrt{3} = \pm (115 + 437\sqrt{3})(2 + \sqrt{3})^n, \tag{8}$$

$$V_n + U_n\sqrt{3} = \pm (-115 + 437\sqrt{3})(2 + \sqrt{3})^n, \tag{9}$$

$$V_n + U_n\sqrt{3} = \pm (269 + 459\sqrt{3})(2 + \sqrt{3})^n, \tag{10}$$

and

$$V_n + U_n\sqrt{3} = \pm (-269 + 459\sqrt{3})(2 + \sqrt{3})^n, \tag{11}$$

where n is zero or an integer.

(i) Considering V_n, U_n satisfying (8) or (9), we easily see that $(V_n, U_n) = 23$ and therefore (8) and (9) will not lead to primitive solutions of the equation (2).

(ii) Considering V_n, U_n given by :

$$V_n + U_n\sqrt{3} = (269 + 459\sqrt{3})(2 + \sqrt{3})^n,$$

we easily see that the residues of U_n modulo 23 are periodic with respect to n, the length of a period being 11 and the residues of a period being 22, 14, 11, 7, 17, 15, 20, 19, 10, 21, 5. Since all these residues are quadratic non-residues modulo 23, (5) is impossible.

(iii) Considering V_n, U_n given by

$$V_n + U_n\sqrt{3} = -(269 + 459\sqrt{3})(2 + \sqrt{3})^n,$$

we see that the residues of V_n modulo 23 are periodic with respect to n, the length of a period being 11 and the residues of a period being 7, 17, 15, 20, 19, 10, 21, 5 22, 14, 11. Since all these residues are quadratic non-residues modulo 23, (4) is impossible.

(iv) Considering V_n, U_n given by

$$V_n + U_n\sqrt{3} = (-269 + 459\sqrt{3})(2 + \sqrt{3})^n,$$

we see that the residues of U_n modulo 23 are periodic with respect to n, the length of a period being 11 and the residues of a period being 22, 5, 21, 10, 19, 20, 15, 17, 7, 11, 14. Since all these residues are quadratic non-residues modulo 23, (5) is impossible.

(v) From (4) and (5), it is clear that both V_n and U_n must be greater than -662. If V_n, U_n satisfy

$$V_n + U_n\sqrt{3} = -(-269 + 459\sqrt{3})(2 + \sqrt{3})^n,$$

we easily see that either V_n or U_n is less than — 662 for all values of n except $n = 0$, and therefore (4) or (5) is impossible when $n \neq 0$. When $n = 0$, $U_0 = -459$ and therefore (5) is impossible in this case.

Thus when $m = 23$, the equation (2) has no primitive solution.

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