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On the Steady Rotation of Stars

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Abstract : The incompatibility of the secular stability criteria and the steady state criteria in the radiative regions of rotating stars is discussed. Although it may not be possible to achieve a stable steady state over the entire radiative region of a star, it is suggested that the star may be able to set up stable steady rotation with circulation over a large portion of its radiative region in one of two ways ; in the more likely of these the angular velocity will be inversely proportional to the square of the distance from the rotation axis except near the axis.

1. Introduction

It is well known that if the angular velocity distribution is specified in a rotating star, strict radiative equilibrium is, in general, not possible in regions stable to convection (e.g. Mestel[4], Roxburgh[6]). This will result in the setting up of large scale internal circulations, which will in general disturb the angular velocity distribution, and thus, a continuous interaction between rotation and circulation will ensue. So the problem of constructing rotating stellar models leads us to look for steady state solutions, in which there is no interaction between rotation and circulation. However, additional difficulties arise when we note that certain constraints must be imposed on the angular velocity distribution to prevent the onset of secular instability (Goldreich and Schubert[2], Fricke[1], Simon[7]). Hence, it is important to inquire whether it is at all possible to construct models which can be both steady and stable. We first note that we may look for models of rotating stars with or without large scale circulations, but Fricke[1] has shown that models without circulations must suffer from secular instability. Hence, we shall focus our attention on models with circulations. We shall confine our attention to axisymmetric models with no magnetic fields. The possibility of constructing stable steady models has been previously considered and it can be shown that stable models cannot be constructed in which the rotation and circulation are everywhere steady in a radiative region (Maheswaran[3]). Strittmatter[9] and Roxburgh[6] have also noted the impossibility of constructing stable steady models. The purpose of this communication is to consider the positive side of this question, i.e. how do we handle this clash between the stability and the steady state criteria.

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2. The Equations

The condition for steady circulation in a non magnetic star is (Mestel[4])

$$\mathbf{v}.\ \nabla\ (\Omega\bar{w}^2) = 0 \tag{1}$$

where v is the circulation velocity, Ω the angular velocity and $\bar{\omega}$ the axial distance. Goldreich and Schubert, and Fricke give the conditions for secular stability (hereinafter referred to as the GSF criteria) in cylindrical polars $(\bar{\omega}, \phi, z)$ as

$$\frac{\partial}{\partial\bar{\omega}}(\Omega\,\bar{\omega}^2) \geqslant 0 \tag{2}$$

and

$$\frac{\partial\Omega}{\partial z} = 0 \tag{3}$$

Simon also obtains equation (2) but replaces equation (3) by a different criterion.

Finally, we must include the equation of conservation of mass in a steady state, which is

$$\operatorname{div}(\rho\,\mathbf{v}) = 0. \tag{4}$$

3. Discussion

Using equations (1), (2) and (4) it is possible to show that steady stable rotation with circulation may exist (i) in regions which are sandwiched between unsteady or unstable regions, which act as sources and sinks to the stable steady regions or (ii) if the rotation law, to be considered presently, has a singularity on the axis of rotation (Maheswaran[3]). Neither equation (3) nor Šimon's modification of it influences these conclusions.

In order to faciliate further discussion we note briefly the nature of the clash between the steady state and the stability criteria, restricting our attention to the GSF criteria. Using cylindrical polar coordinates equation (1) may be written

$$v_{\bar{\omega}}\frac{\partial}{\partial\bar{\omega}}(\Omega\bar{\omega}^2) + v_z\frac{\partial}{\partial z}(\Omega\bar{\omega}^2) = 0\,. \tag{5}$$

Using equation (3) we may simplify this to

$$v_{\bar{\omega}}\frac{\partial}{\partial\bar{\omega}}(\Omega\bar{\omega}^2) = 0\,. \tag{6}$$

We now consider separately the two cases arising from equation (2).

(*a*) If

$$\frac{\partial}{\partial\bar{\omega}}(\Omega\bar{\omega}^2) > 0, \tag{7}$$

equation (6) yields

$$v_{\bar{\omega}} = 0, \tag{8}$$

which requires the flow to take place along lines parallel to the axis of rotation. This clearly cannot satisfy the condition of mass conservation in equation (4), unless there exist surrounding regions which supply and remove material.

(*b*) If

$$\frac{\partial}{\partial\bar{\omega}}(\Omega\bar{\omega}^2) = 0 \tag{9}$$

$v_{\bar{\omega}}$ need not vanish and steady circulation will be possible. However, in this case

$$\Omega \propto \frac{1}{\bar{\omega}^2}, \tag{10}$$

which implies a singularity for Ω on the axis. Again, this problem may be overcome by supposing that

$$\Omega = \Omega_0 \frac{\bar{\omega}_0^{\,2}}{\bar{\omega}_2} \qquad \text{in } \bar{\omega} \geqslant \text{some } \bar{\omega}_0,$$

where Ω_0 is constant, and that in the region $\bar{\omega} < \bar{\omega}_0$ the meridional motion is turbulent, the angular velocity, perhaps being constant, with material flowing in and out of this region to link up with the circulation outside.

If instead of the GSF criteria we use Simon's criteria, when equation (10) holds the discussion of case (b) remains unchanged. On the other hand if equation (9) is the case, equations (1) and (2) imply that it will not be possible to have closed strealines of circulation in a bounded region of a meridian plane (Maheswaran[3]). Thus, to satisfy equation (4) steady motion will be possible only in a region which is surrounded by regions which act as sources and sinks. The only difference here being that $v_{\bar{\omega}}$ does not necessarily vanish, and so the streamlines of the meridional flow need not be parallel to the axis of rotation.

4. Conclusions

The preceding discussion indicates that, though the clash between the steady state criteria and the stability criteria prevents the entire radiative region of a star from reaching a state of stable steady rotation with circulation, it will be possible for a star to achieve such a state over a large portion of its radiative region in one of two ways. These correspond to the cases (a) and (b) of § 3 and we shall refer to them as type I and type II steady states respectively.

In the type I steady state the flow takes place from one turbulent or unstable region into another through the steady stable region, in which there are no closed streamlines. If the GSF criteria are the operative ones, then the flow must be parallel to the axis of rotation. In the case of Simon's criteria this may be relaxed provided lines parallel to the equatorial plane do not cut a streamline more than once. Since we know that the circulation pattern is determined by the angular velocity distribution pattern (Sweet[10]), in order to achieve the type I steady state, the star must seek that distribution of Ω which will drive the permitted motion. Assuming that a star is in fact capable of searching out such a state, we note that the theoretical problem of constructing such models will be considerable since it would require us to work the problem backwards, i.e. to guess correctly the angular velocity distribution.

The type II steady state is the more promising candidate. For, if the circulation is truly large scale, it is likely that during the period in which circulation completes a few circuits, the redistribution of angular velocity would lead to a steady state with $\Omega \propto 1/\bar{\omega}^2$ over most of the radiative zone together with a region near the axis which is in a state of turbulence and possibily in uniform rotation. In this case the meridional motion will look more like a circulation and in order to ensure continuity the streamlines must link up with the motion in the turbulent zone. Further, we might note that, on the theoretical side, the construction of models is considerably simplified since Sweet's method may now be employed.

A further point worth noting is that, in an axisymmetric rotating star, the nature of the forces present will be such that equatorial symmetry will also exist. Hence, there will be no motion from one hemisphere in to the other. Thus, in order to achieve a steady state of Type I, it will be necessary for a turbulent zone to be set up along the equatorial plane.

Figures 1 to 4 show the division of a meridional quadrant into the various zones for type I and type II steady states in lower and upper main sequence stars.

It is interesting to note that the possible existence of a turbulent outer zone may also be arrived at from an attempt to solve the $1/\rho$ singularity in the circulation speed near the surface of a star as shown by Smith[8] and Osaki.[5]

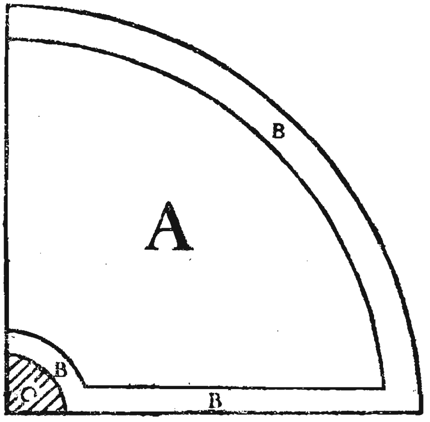

Fig. 1 : Type I steady state in an upper main sequence star. The figure shows the division of a quadrant of a meridian section into the various zones. A = portion of the radiative region in steady rotation and meridional motion. B = portion of radiative region in turbulent motion. C = convective region.

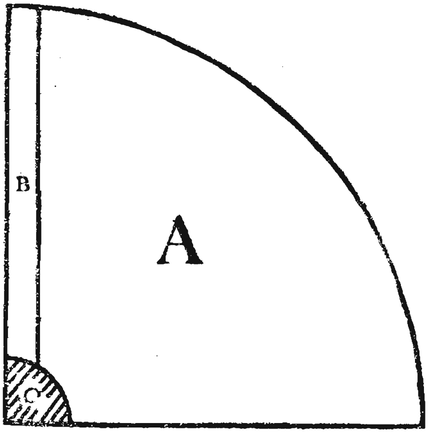

Fig. 2 : Type II steady state in an upper main sequence star. A, B, C are as defined in fig. 1.

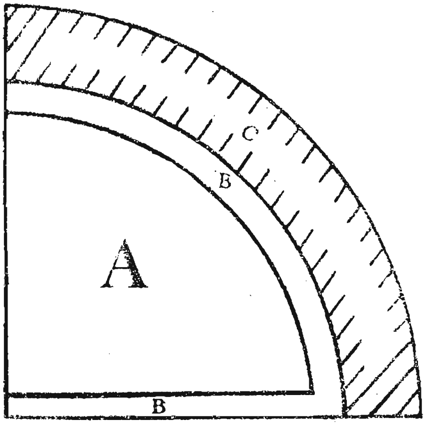

Fig. 3 : Type I steady state in a lower main sequence star. A, B, C are as defined in fig. 1.

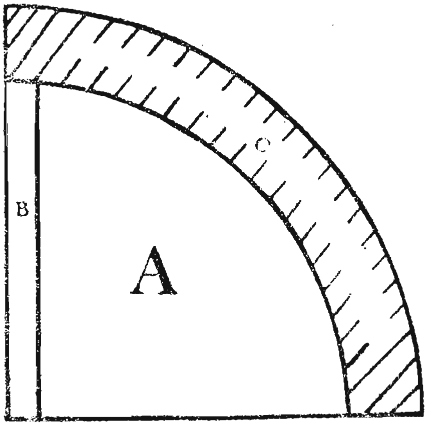

Fig. 4 : Type II steady state in a lower main sequence star. A, B, C are as defined in fig. 1.

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