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Conditions for Steady Circulation in Rotating Magnetic Stars with Finite Electrical Conductivity

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Abstract : It is shown that the conditions for steady circulation derived for perfectly conducting rotating magnetic stars by Chandrasekhar and Mestel may be extended to fluids with finite electrical conductivity provided that the magnetic flux density belongs to the class of solutions of the vector wave equation. When the field becomes force-free it is shown that the stellar fluid and the electric currents must flow along the same streamlines in a meridian plane, and that in the case of slow meridian circulations in stars a steady state might not be achieved with a force-free field unless the azimuthal component of the field is much stronger than the meridional component.

1. Introduction

The study of stellar structure is difficult when rotation and magnetic fields are present. It becomes complicated when large scale circulations are set up in the interior, which may react back on the rotation and magnetic fields that drive them. The easiest approach to this problem is to study structures in which a steady state has been reached. In such a state we would expect the rotation, circulation and magnetic field to be related to each other in some way. The conditions for steady circulation in an axisymmetric star, when its material is treated as a perfect electrical conductor have been obtained by Chandrasekhar[1] and Mestel.[2] The basic equations for a perfectly conducting system in a steady state are :

$$\text{curl}\ (\mathbf{v} \wedge \mathbf{B}) = 0 \qquad (1)$$

$$\text{div}\ \mathbf{B} = 0 \qquad (2)$$

and

$$\text{div}\ (\rho \mathbf{v}) = 0 \qquad (3)$$

where **B** denotes the flux density, ρ the material density and **v** the velocity. If we denote the meridional and azimuthal components using subscripts *m* and *a* respectively, the steady state conditions in the axisymmetric case may be written (Mestel[4]) in Gaussian units as

$$\mathbf{v}_m = k\,\mathbf{B}_m, \qquad (4)$$

$$\Omega - \frac{kB_a}{\varpi} = \alpha, \qquad (5)$$

$$\rho k = \frac{\rho v_m}{B_m} = \gamma, \qquad (6)$$

and

$$-\frac{\varpi B_a}{4\pi\mu} + \rho k \Omega \varpi^2 = -\frac{\beta}{4\pi}, \tag{7}$$

where Ω is the angular velocity, ϖ the axial distance, k a scalar, μ the permeability and α, β, γ are constants on the streamlines of circulation. Equations (4), (5) and (6) follow from equations (1), (2) and (3). Equation (7) is derived from the azimuthal component of the equation of motion, which says that the convection of angular momentum by the circulation is balanced by its transport by magnetic stresses. We note that for consistency B_a should not vanish.

We note that, though perfect electrical conductivity is a good first approximation for stars, the actual electrical conductivities are such that the magnetic Reynolds number Rm is finite and not very large when we consider the Eddington-Vogt circulations (Maheswaran[3]). Hence, it would be useful to consider the extension of the Chandrasekhar-Mestel problem for stars where the electrical conductivity is finite. In this paper we confine our attention to axisymmetric stars with constant conductivity.

2. The Equations

The Magnetohydrodynamic induction equation, in Gaussian units, reads

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl}\,(\mathbf{v} \wedge \mathbf{B}) - \text{curl}\,(\eta\ \text{curl}\,\mathbf{B}) \tag{8}$$

where $\eta = c^2/4\pi\mu\sigma$ is the magnetic resistivity ; c being the speed of light and σ the electrical conductivity.

We shall consider systems in which η is constant and the magnetic fields decays like exp $(-\eta t/l^2)$, where l denotes some decay length scale. Writing

$$\mathbf{B} = \mathbf{B}_0 \exp\,(-\eta t/l^2) \tag{9}$$

in equation (8) we obtain

$$\text{curl}\,(\mathbf{v} \wedge \mathbf{B}_0) = \eta\,(\text{curl curl}\,\mathbf{B}_0 - \frac{1}{l^2}\,\mathbf{B}_0). \tag{10}$$

We shall refer to the state in which $\mathbf{v}$ is steady and $\mathbf{B}$ decays as described by equation (9) as the quasi-steady state.

The problem of finding general solutions of equation (10) is a difficult one, but a particular solution which is of interest may be obtained easily. We shall consider a system in which equation (10) reduces to

$$\text{curl}\,(\mathbf{v} \wedge \mathbf{B}_0) = 0 = \text{curl curl}\,\mathbf{B}_0 - \frac{1}{l^2}\,\mathbf{B}_0 \tag{11}$$

This gives us

$$\nabla^2 \mathbf{B}_o + \frac{1}{l^2} \mathbf{B}_o = 0 \tag{12}$$

and

$$\mathbf{v} \wedge \mathbf{B}_o = \nabla f, \tag{13}$$

where f is a scalar function.

Equation (12) states that $\mathbf{B}_o$ is a solution of the vector wave equation. Hence, the most general form of $\mathbf{B}_o$ may be written (e.g. vide Ferraro and Plumpton[2], p. 62)

$$\mathbf{B}_o = \lambda_1 \operatorname{curl} (\mathbf{a} \psi) + \lambda_2 \, l \operatorname{curl} \operatorname{curl} (\mathbf{a} \psi) \tag{14}$$

where $\mathbf{a}$ is a constant unit vector, ψ is a scalar satisfying the wave equation

$$\nabla^2 \psi + \frac{1}{l^2} \psi = 0$$

and λ_1, λ_2, are arbitrary constants.

We shall now investigate the conditions for steady circulation that will result from equations (2), (3), (12) and (13).

3. Conditions for Steady Circulation

In the case of a three dimensional system it is not easy to obtain conditions any simpler than those expressed in equations (12) and (13). However, considerable simplification is possible when we look at axisymmetric systems.

3.1. Axisymmetric systems

In an axisymmetric system equation (13) simplifies to

$$\mathbf{v}_m \wedge \mathbf{B}_{om} = 0 \tag{15}$$

and

$$\mathbf{v}_a \wedge \mathbf{B}_{om} + \mathbf{v}_m \wedge \mathbf{B}_{oa} = \nabla f \tag{16}$$

These equations are the same as for $\mathbf{B}$ in the perfect conductor case and therefore equations (15) and (16) together with equations (2) and (3) yield (vide Mestel[4]).

$$\mathbf{v}_m = k\mathbf{B}_{om} \tag{17}$$

$$\Omega \cdot \frac{kB_{oa}}{\varpi} = \alpha \tag{18}$$

and

$$\rho k = \frac{\rho v_m}{B_{om}} = \gamma \tag{19}$$

where α, γ are constant on the streamlines of circulation and k is a scalar.

If we take the equation of motion in the azimuthal sense we have

$$\left(\frac{\text{curl } \mathbf{B} \wedge \mathbf{B}}{4\pi\mu}\right)_a = \rho \frac{1}{\varpi} \mathbf{v}_m \cdot \nabla (\Omega \varpi^2) \tag{20}$$

which may be written

$$\exp(-2\eta t/l^2)\left(\frac{\text{curl } \mathbf{B}_o \wedge \mathbf{B}_o}{4\pi\mu}\right)_a = \frac{\rho}{\varpi} \mathbf{v}_m \cdot \nabla (\Omega\varpi^2). \tag{21}$$

The presence of the time dependent term complicates the problem. However, if we are interested only in a time scale short compared with the decay time scale i.e $t << l^2/\eta$, we can approximate equation (21) to

$$\varpi\left(\frac{\text{curl } \mathbf{B}_0 \wedge \mathbf{B}_0}{4\pi\mu}\right)_a = \rho\mathbf{v}_m \cdot \nabla (\Omega\varpi^2). \tag{22}$$

As in the perfect conductor case we assume that $\mathbf{B}_{oa}$ does not vanish. Further, we shall assume that the field is not force-free, for otherwise the left hand side of equation (22) will vanish. We shall consider the force-free case separately in the ensuing section. Equation (22) together with equations (17) and (19) yield (as in Mestel[4])

$$-\frac{\varpi B_{oa}}{4\pi\mu} + \rho k\,\Omega\,\varpi^2 = -\beta/4\pi. \tag{23}$$

Hence, a set of conditions for steady circulation over time scales short compared with the decay time scale are given by equations (17), (18), (19), (23) and (12) together with equation (9); i.e. a set of conditions for the finite conductor case may be obtained by replacing **B** by $\mathbf{B}_o$ in the conditions for the perfect conductor case with the additional condition that $\mathbf{B}_o$ should belong to the class of solutions of the vector wave equation.

3.2. Force-free fields

In the preceding section we noted that if the field was force-free, the left hand side of equation (22) vanishes. Also, we know that a class of solutions of equations (12) for $\mathbf{B}_o$ is that of the force-free fields, which will be given by

$$\text{curl } \mathbf{B}_o = \frac{1}{l} \mathbf{B}_o. \tag{24}$$

It is known that a field cannot be everywhere force-free in an isolated system in which the field vanishes at the surface (e.g. vide Ferraro & Plumpton[2]). However, we may apply our results to a portion of a larger system (e.g. like the radiative region of an axisymmetric star).

We shall now discuss the consequences of supposing that $\mathbf{B}_o$ satisfies (24). The electric current vector $\mathbf{j}$ is given by

$$\mathbf{j} = \frac{1}{4\pi\mu} \text{ curl } \mathbf{B} \tag{25}$$

If we put $\mathbf{j} = \mathbf{j}_o \exp(-\eta t/l^2)$ we have from equations (24) and (25) that

$$\mathbf{j}_o = \mathbf{B}_o/4\pi\mu l. \tag{26}$$

Hence, we may write, using equations (17), (18) and (19), which remain unaltered,

$$\mathbf{v}_m = k'\mathbf{j}_{om}, \tag{27}$$

$$\Omega - \frac{k'}{\varpi} j_{oa} = \alpha' \tag{28}$$

and

$$\rho k' = \frac{\rho v_m}{J_{om}} = \gamma', \tag{29}$$

where k' is a scalar and α', γ' are constants along the streamlines. Equations (28) and (29) together yield

$$\mathbf{v} = k'\mathbf{j}_o + \alpha' \varpi \, \mathbf{e}_\phi \tag{30}$$

where $\mathbf{e}_\phi$ is the unit azimuthal vector. This equation tells us that the motion consists of an arbitrary uniform rotation of each poloidal loop superimposed on a velocity $k'\mathbf{j}_o$.

When we come to the azimuthal component of the equations of motion we find that the earlier condition (23) is altered. The equation of motion now reads

$$\mathbf{v}_m \cdot \nabla(\Omega \varpi^2) = 0, \tag{31}$$

which requires that either $\mathbf{v}_m$ vanish or the angular momentum per unit mass be constant along the streamlines. i.e.

$$v_m = 0 \quad \text{or} \quad \Omega\varpi^2 = \beta', \tag{32}$$

where β' is constant on streamlines.

Hence, a set of conditions that may be used to obtain systems with steady circulation is given by equations (27), (28), (29), (32) and (24) together with equation (9).

4. Discussion

Since the form of the steady state equations discussed here are the same as those for the perfect conductor case their consequences will remain essentially the same except for the restrictions on the choice of the magnetic field **B**. Hence, the discussion provided by Mestel[4,5] will be valid in this special case too.

The following cases are worth noting here :

(i) $v_m = 0$.

When $v_m = o$ we have no circulation. So $k = 0 = k'$ and the angular velocity will be constant on the field lines which is Ferraro's law of isorotation.

(ii) $v_m << v_a = \Omega\bar{\omega}$

This is true for slow circulations (as in the case of the Eddington-Vogt circulations in stars). In this situation

(*a*) If $B_a \lesssim B_m$, we have from either equations (17) and (18) or equations (27) and (28) that

$$\Omega = \text{constant on streamlines.} \tag{33}$$

Now, if the field is not force-free this will be possible. However, if the field is force-free, equations (31) and (33) will be inconsistent. So, in this case, in order to maintain a steady state the field must not be force-free. We might note that this applies equally to the perfect conductor case.

(*b*) If $B_a >> B_m$ such that $v_m B_a \sim v_a B_m$, then equations (18) and (28) retain their forms and there is no apparent inconsistency.

Hence, we might conclude that in a case where the circulation speed is small compared with the rotation speed, the field must either be not force-free or, if it is force-free then $B_a/B_m \sim v_a/v_m$.

What we have shown in this paper is that the equations of Chandrasekhar and Mestel for steady circulation in an axisymmetric perfectly conducting rotating magnetic star may be extended to a fluid of finite conductivity with the additional condition that the magnetic field **B** belongs to the class of solutions of the vector wave equation and has a uniform exponential decay. However, though these equations are convenient for application to models, they represent a situation in which a characteristic feature of material with finite conductivity is absent. Whereas in the perfect conductor the field lines and the streamlines of circulation must coincide, we know that in material of finite conductivity the field lines may cross the streamlines. Therefore, it is likely that this set of conditions will be relevant only for special cases of rotating magnetic stars. Besides, we should note that in the radiative regions of real stars the electrical conductivity will change with position.

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