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RESEARCH ARTICLE

Statistics: distribution theory

A new multivariate transmuted family of distributions: theory and application for modelling of daily world COVID-19 cases

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Abstract: Multivariate distributions are helpful in the simultaneous modeling of several dependent random variables. The development of a unique multivariate distribution has been a difficult task and different multivariate versions of the same distribution are available. The need is, therefore, to suggest a method of obtaining a multivariate distribution from the univariate marginals. In this paper, we have proposed a new method of generating the multivariate families of distributions when information on univariate marginals is available. Specifically, we have proposed a multivariate family of distributions which provides a univariate transmuted family of distributions as marginal. The proposed family is a re-parameterization of the Cambanis (1977) family. Some properties of the proposed family of distributions have been studied. These properties include marginal and joint marginal distributions, conditional distributions, and marginal and conditional moments. We have also obtained the dependence measures alongside the maximum likelihood estimation of the parameters. The proposed multivariate family of distributions is studied for the Weibull baseline distributions giving rise to the multivariate transmuted Weibull (MTW) distribution. Real data application of the proposed *MTW* distribution is given in the context of modeling the daily COVID-19 cases of the World. It is observed that the proposed *MTW* distribution is a suitable fit for the joint modeling of the COVID-19 data.

Keywords: Dependence functions, maximum likelihood estimation, multivariate *T–X* family of distributions, transmuted distributions, Weibull distribution.

INTRODUCTION

Probability distributions have been widely used in several areas of life. Certain situations arise where the standard probability models are not capable of capturing complex behaviour of the data and hence some extensions are required. Numerous methods are available in the literature to extend any baseline distribution. One such method of extending the univariate distribution is the transmuted family of distributions proposed by Shaw and Buckley (2007). The cumulative distribution function (cdf) of the transmuted family of distributions is

$$F(x) = G(x)\left[1+\lambda-\lambda G(x)\right];\ -1\le\lambda\le 1$$

where $G(x)$ is the *cdf* of any baseline distribution and λ is the transmutation parameter. The transmuted family of distributions is useful to obtain the transmuted version of any baseline distribution such that the resulting distribution has much wider applicability as compared with the baseline distribution. More details about the transmuted distributions can be found in Rahman *et al*. (2020).

The transmuted family of distribution can be obtained as a member of the *T–X* family of distributions proposed by Alzaatreh *et al*. (2013). The transmuted family of distributions has been studied by several authors for different baseline distributions. Some examples include the transmuted-G family of distributions by Nofal *et al*. (2016), the Kumaraswamy transmuted-G family of distributions by Afify *et al*. (2016), the beta transmuted-H family by Afify *et al*. (2017) and the T-transmuted X family of distributions by Jayakumar and Babu (2017). A comprehensive review of developments in transmuted distributions has been given in Rahman *et al*. (2020).

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The development in the area of bivariate and multivariate families of distributions is relatively challenging and not many methods are available to obtain the bivariate or multivariate distribution from the given marginals. A classical method to obtain the bivariate distribution, from the given univariate marginals, has been proposed by Gumbel (1960) as is known as the Gumbel bivariate distribution. The joint *cdf* of the Gumbel bivariate distribution is

$$F(x,y) = G(x)G(y)\left[1+\gamma\{1-G(x)\}\{1-G(y)\}\right];$$
$$0 \le \gamma \le 1$$

where $G(x)$ and $G(y)$ are any marginal *cdf*'s and γ is an association parameter. This family has been studied by various authors. For example, the bivariate Kumaraswamy distribution has been studied by Barreto-Souza and Lemonte (2013) and the bivariate Pareto distributions by Sankaran *et al*. (2014) among others. Recently, Sarabia *et al*. (2014) have extended the *Beta – G* family of distributions to the bivariate case by using bivariate beta distribution as baseline distribution. Darwish *et al*. (2021) have used the bivariate *T–X* family of distributions to propose a new bivariate transmuted family of distributions. The proposed family can be used with any baseline distribution. This family has opened the door for development of new bivariate distributions by using any baseline distributions.

Various complex situations arise where the joint modeling of several variables is required and in such cases the multivariate distributions are required. The multivariate families of distributions have not been investigated. Some common multivariate distributions include generalization of the power exponential family of distributions, by Gomez *et al*. (1998) and family of multivariate generalized *t* distributions by Arslan (2004). The main aim of this study is to propose a new family of distributions which generate a multivariate distribution from the given marginals. The proposed family will be named as the *multivariate transmuted family of distributions*. This new family will be suitable for any baseline distributions and will be useful in modelling joint and complex phenomena.

MATERIALS AND METHODS

The methodology in this paper is primarily based upon the transmuted distributions and the *T–X* family of distributions. The transmuted distributions are briefly discussed in the introduction. The *T–X* family of distributions has been proposed by Alzaatreh *et al*. (2013). The *cdf* of the proposed family is

$$F_{T-X}(x) = \int_a^{W[G(x)]} r(t)\,\mathrm{d}t = R[W\{G(x)\}], x \in \Re, \quad ..(1)$$

where $r(t)$ is any probability distribution defined over $[a,b]$, $W[G(x)]$ is any differentiable function of $G(x)$ such that $W(0) \to a$ and $W(1) \to b$. The probability density function (*pdf*) corresponding to (1) is

$$f_{T-X}(x) = \left[\frac{d}{dx}W\{G(x)\}\right] r\left[W\{G(x)\}\right], x \in \Re \cdot$$

The *T–X* family of distributions can be used to propose a new distribution by using a suitable $r(t)$.

The transmuted family of distributions, proposed by Shaw and Buckley (2007), can be obtained from the *T–X* family of distributions by using a suitable $r(t)$ with support on $[0,1]$ in (1) and $W[G(x)] = G(x)$. Alizadeh *et al*. (2017) have shown that the transmuted family of distribution can be obtained by using $r(t) = 1+\lambda-2\lambda t$ in (1).

In this paper, we focus on extending the transmuted family of distributions to the multivariate case. It is, therefore, suitable to present the bivariate *T–X* family of distributions and then extend it to propose a new multivariate transmuted family of distributions. The *cdf* of the bivariate *T–X* family of distributions is

$$F_{X_1,X_2}(x_1,x_2) = \int_{a_2}^{W[G_2(x_2)]}\int_{a_1}^{W[G_1(x_1)]} r(u_1,u_2)\,du_1 du_2, \quad ...(2)$$

with the usual properties of $W[G(x_1)]$, and $W[G(x_2)]$ and $r(u_1,u_2)$ is any bivariate distribution with suitable support for random variables U and U_2.

The multivariate extension of (2) is immediately written as

$$F_{\mathbf{x}}(\mathbf{x}) = \int_{a_p}^{W[G_p(x_p)]}\cdots\int_{a_1}^{W[G_1(x_1)]} r(u_1,u_2,\ldots,u_p)\,du_1\cdots du_p, \quad ...(3)$$

where $\mathbf{x} = (X_1, X_2, \ldots, X_p)$ is a *p*–vector of random variables and $r(u_1,u_2,\ldots,u_p)$ is any *p*–variate density function with support over $[a_1,b_1]\times\cdots\times[a_p,b_p]$.

A simpler version of (3) is obtained when the random vector $\mathbf{u} = \left(u_1, u_2, \ldots, u_p\right)$ has support over $\left[0,1\right]^p$ and is given as

$$F_{\mathbf{x}}\left(\mathbf{x}\right) = \int_0^{G_p^{\phi_p}\left(x_p\right)} \cdots \int_0^{G_1^{\phi_1}\left(x_1\right)} r\left(u_1, u_2, \ldots, u_p\right) du_1 \cdots du_p \,, \quad \text{...(4)}$$

where $\left(\phi_1, \phi_2, \ldots, \phi_p\right) > 0$.

RESULTS AND DISCUSSION

In this section, a new multivariate family of distributions has been proposed which provides the univariate transmuted family of distributions as marginals. The properties of the proposed families are studied. The new multivariate family of distributions is proposed in the following sub-section.

A new multivariate family of distributions

In this subsection, a new multivariate family of distributions has been proposed. This multivariate family has been proposed by using (4) with a suitable choice of $r\left(u_1, u_2, \ldots, u_p\right)$. The *cdf* of the multivariate transmuted family of distributions can be obtained by using

$r\left(u_1, u_2, \ldots, u_p\right) = 1 + \sum_{i=1}^{p} \lambda_i \left(1 - 2u_i\right) + \lambda_{p+1}\left(p - 2\sum_{i=1}^{p} u_i\right)$, and $\phi_1 = \phi_2 = \cdots = \phi_p = 1$ in (4) and is given as

$$F_{\mathbf{x}}\left(\mathbf{x}\right) = \int_0^{G_p\left(x_p\right)} \cdots \int_0^{G_1\left(x_1\right)} \left[1 + \sum_{i=1}^{p} \lambda_i \left(1 - 2u_i\right) + \lambda_{p+1}\left(p - 2\sum_{i=1}^{p} u_i\right)\right] du_1 \cdots du_p \,.$$

Solving the above multiple integral, the *cdf* of multivariate family of distributions is

$$F_{\mathbf{x}}\left(\mathbf{x}\right) = \left\{\prod_{i=1}^{p} G_i\left(x_i\right)\right\}\left[1 + \sum_{i=1}^{p}\left(\lambda_i + \lambda_{p+1}\right)\left\{1 - G_i\left(x_i\right)\right\}\right]; \quad \mathbf{x} \in \Re^p, \quad \text{...(5)}$$

where $G_i\left(x_i\right)$ is the marginal *cdf* of the *i*th random variable X_i, and $\left(\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}\right)$ are transmutation parameters such that $\left(\lambda_i + \lambda_{p+1}\right) \in \left[-1, 1\right]$ for $i = 1, 2, \ldots, p$ and $-1 \le \sum_{i=1}^{p} \lambda_i + p\lambda_{p+1} \le 1$. The proposed family will be named as the multivariate transmuted family of distributions and is a re-parameterization of the Cambanis (1977) family.

The density function corresponding to (5) is

$$f_{\mathbf{x}}\left(\mathbf{x}\right) = \left\{\prod_{i=1}^{p} g_i\left(x_i\right)\right\}\left[1 + \sum_{i=1}^{p}\left(\lambda_i + \lambda_{p+1}\right)\left\{1 - 2G_i\left(x_i\right)\right\}\right]; \quad ; \mathbf{x} \in \Re^p, \quad \text{...(6)}$$

where $g_i\left(x_i\right)$ is density function corresponding to $G_i\left(x_i\right)$. It is to be noted that the transmuted family of distributions, proposed by Shaw and Buckley (2007), turns out to be a special case of (5) for $p = 1$ and $\lambda_{p+1} = 0$. The bivariate transmuted family of distributions, proposed by Darwish *et al*. (2021), appears as special case of (5) for $p = 2$.

The proposed multivariate transmuted family of distributions can be used to obtain multivariate distributions from the univariate marginals. In the following section, we will give some properties of the new proposed multivariate transmuted family of distributions.

Properties of the multivariate transmuted family of distributions

This sub-section contains some important properties of the proposed multivariate family of distributions. These include the marginal distributions, conditional distributions, conditional moments and multivariate dependence measures. We will also give an estimation of the unknown model parameters.

The marginal distributions

In this subsection, we will obtain the univariate marginal distribution of a single variable, the bivariate marginal distributions of two random variables and the joint marginal distribution of a subset. The marginal distribution function of *i*th variable is obtained as

$$F_{X_i}\left(x_i\right) = G_i\left(x_i\right)\left[\text{ü} + \left(\lambda_i + \lambda_{p+1}\right)\left\{\ - G_i\left(x_i\right)\right\}\right] \quad x_i \in \Re, \quad \text{...(7)}$$

which is a transmuted family of distribution with transmutation parameter $\lambda_i + \lambda_{p+1}$. The marginal density function of *i*th random variable is immediately written as

$$f_{X_i}\left(x_i\right) = g_i\left(x_i\right)\left[1 + \left(\lambda_i + \lambda_{p+1}\right)\left\{1 - 2G_i\left(x_i\right)\right\}\right]; \; x_i \in \Re, \quad \text{...(8)}$$

where $g_i\left(x_i\right)$ is the density function of ith random variable. The joint marginal distribution of two random variables $\left(X_i, X_m\right)$ is obtained by using $G_j\left(x_j\right)=1$, for $j=1,2,\ldots,p$; $j\neq\left(i,m\right)$ in (5) and is given as

$$F_{X_i,X_m}\left(x_i,x_m\right)=G_i\left(x_i\right)G_m\left(x_m\right)\Big[1+\left(\lambda_i+\lambda_{p+1}\right)\left\{1-G_i\left(x_i\right)\right\}+\left(\lambda_m+\lambda_{p+1}\right)\left\{1-G_m\left(x_m\right)\right\}\Big], \qquad \text{...(9)}$$

for $\left(x_i,x_m\right)\in\mathbb{R}^2$ with $G_i\left(x_i\right)$ and $G_m\left(x_m\right)$ being the *cdf*'s of ith and mth random variables. The joint *cdf*, given in (9) is same as given in Darwish *et al*. (2021). The joint bivariate density function of two random variables is

$$f_{X_i,X_m}\left(x_i,x_m\right)=g_i\left(x_i\right)g_m\left(x_m\right)\Big[1+\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}+\left(\lambda_m+\lambda_{p+1}\right)\left\{1-2G_m\left(x_m\right)\right\}\Big] \qquad \text{...(10)}$$

The joint marginal distribution of a subset of variables, say $\mathbf{x}_1=\left(X_1,X_2,\ldots,X_t\right)$, is given as

$$f_{\mathbf{x}_1}\left(\mathbf{x}_1\right)=\left\{\prod_{i=1}^{t}g_i\left(x_i\right)\right\}\Big[1+\sum_{i=1}^{t}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}\Big];\ \mathbf{x}_1\in\Re^t, \qquad \text{...(11)}$$

which is also a multivariate transmuted family of distribution.

Conditional distributions

We can study several conditional distributions in case of a multivariate distribution and in the following we have obtained some conditional distributions for the multivariate transmuted family of distributions. The conditional density function of a single variable for given information of the other variables is defined as

$$f\left(x_i\mid\mathbf{x}_{(i)}\right)=\frac{f\left(x_i,\mathbf{x}_{(i)}\right)}{f\left(\mathbf{x}_{(i)}\right)},$$

where $\mathbf{x}_{(i)}=\left(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_p\right)$. The conditional distribution of X_i given the information of all other variables for the multivariate transmuted distribution is

$$f\left(x_i\mid\mathbf{x}_{(i)}\right)=\frac{g_i\left(x_i\right)}{\Delta_i\left(\mathbf{x}_{(i)}\right)}\Big[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}\Big];\ x_i\in\Re\ ;\ \mathbf{x}_{(i)}\in\Re^{p-1}, \qquad \text{...(12)}$$

where $\Delta_i\left(\mathbf{x}_{(i)}\right)=\Big[1+\sum_{j\neq i=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{1-2G_j\left(x_j\right)\right\}\Big]$.

The joint conditional distribution of two random variables X_i and X_m given the information of other variables is obtained by using

$$f\left(x_i,\mathrm{x}_m\mid\mathbf{x}_{(i,m)}\right)=\frac{f\left(x_i,\mathrm{x}_m,\mathbf{x}_{(i,m)}\right)}{f\left(\mathbf{x}_{(i,m)}\right)},$$

where $\mathbf{x}_{(i,m)}=\left(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_{m-1},X_{m+1},\ldots,X_p\right)$. The joint conditional distribution of two random variables for a multivariate transmuted family of distributions is

$$f\left(x_i,x_m\mid\mathbf{x}_{(i,m)}\right)=\frac{g_i\left(x_i\right)g_m\left(x_m\right)}{\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)}\Big[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}\Big], \qquad \text{...(13)}$$

where $\left(x_i,x_m\right)\in\Re^2$, $\mathbf{x}_{(i,m)}\in\Re^{p-2}$ and

$$\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)=\Big[1+\sum_{j\neq(i,m)=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{1-2G_j\left(x_j\right)\right\}\Big].$$

Proceeding in a similar way, it is easy to show that the conditional density function of an one subset of variables, say $\mathbf{x}_1$, given the information of an other set of variables, say $\mathbf{x}_2$, in multivariate transmuted family of distribution is given as

$$f\left(\mathbf{x}_1\mid\mathbf{x}_2\right)=\frac{1}{\Delta_t\left(\mathbf{x}_2\right)}\prod_{i=1}^{t}g_i\left(x_i\right)\Big[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}\Big];\ \mathbf{x}_1\in\Re^{\ddot{u}}\ ;\ \mathbf{x}_2\in\Re^{\ -} \qquad \text{...(14)}$$

where $\Delta_t\left(\mathbf{x}_2\right)=\Big[1+\sum_{j=t+1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{1-2G_j\left(x_j\right)\right\}\Big]$.

The conditional distributions can be studied for any baseline distributions and can be used to compute the conditional moments which are given in the following subsection.

Conditional moments

The single and product moments for univariate and bivariate transmuted family of distributions can be found on the lines given in Darwish *et al*. (2021). We can also obtain various conditional moments by using the multivariate transmuted family of distributions. The rth conditional moment of X_i given the information of other variables for multivariate transmuted family of distributions is obtained by using

$$\mu'^{r}_{X_i|\mathbf{x}_{(i)}}=\int_{-\infty}^{\infty}x_i^r f\left(x_i\mid\mathbf{x}_{(i)}\right)dx_i,$$

which for the conditional distribution in (12) is

$$\mu'^{r}_{x_i|\mathbf{x}_{(i)}} = \frac{1}{\Delta_i\left(\mathbf{x}_{(i)}\right)}\int_{-\infty}^{\infty} x^r g\ (x\ )\Big[1+\sum_{j=1}^{p}\left(\lambda\ +\lambda\ _{+1}\right)$$

$$\left\{1-2G\ (x\ )\right\}\Big]dx\ ,$$

and on simplification we have

$$\mu'^{r}_{x_i|\mathbf{x}_{(i)}} = \frac{1}{\Delta_i\left(\mathbf{x}_{(i)}\right)}\Big[\beta\mu'^{r}_{\ i} - 2\mu'^{r}_{\ i}\sum_{j\neq i=1}^{p}\left(\lambda\ +\lambda\ _{+1}\right)G\ \left(x\ \right)$$

$$-\left(\lambda_i+\lambda_{p+1}\right)\mu'^{r}_{x_i(2:2)}\Big], \qquad \text{...(15)}$$

where $\beta = \ +\sum_{i=}^{p}\lambda_i + p\lambda_{p+}$, $\mu'^{r}_{x_i}$ is rth raw moment of X_i and $\mu'^{r}_{x_i(2:2)}$ is rth moment of larger observation in a sample of size 2 from $G_i\left(x_i\right)$.

The joint conditional moment of X_i and X_m given the information of other variables is obtained by using

$$\mu'^{r,s}_{X_i,X_m|\mathbf{x}_{(i,m)}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^r x_m^s f\left(x_i, x_m \mid \mathbf{x}_{(i,m)}\right)dx_i dx_m\ .$$

Using the joint conditional distribution of X_i and X_m from (13), we have

$$\mu'^{r,s}_{X_i,X_m|\mathbf{x}_{(i,m)}} = \frac{1}{\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^r x_m^s g_i\left(x_i\right)g_m\left(x_m\right)$$

$\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}\right]dx_i dx_m$. the above integral, the joint conditional moment of X_i and X_m given the information of other variables is

$$\mu'^{r,s}_{X_i,X_m|\mathbf{x}_{(i,m)}} = \frac{1}{\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)}\Big[\beta\mu'^{r}_{x_i}\mu'^{s}_{x_m} - 2\mu'^{r}_{x_i}\mu'^{s}_{x_m}\sum_{j\neq(i,m)=1}^{p}$$

$$\left(\lambda_j+\lambda_{p+1}\right)G_j\left(x_j\right) - \left(\lambda_i+\lambda_{p+1}\right)$$

$$\times\mu'^{r}_{x_i(2:2)}\mu'^{s}_{x_m} - \left(\lambda_m+\lambda_{p+1}\right)\mu'^{r}_{x_i}\mu'^{s}_{x_m(2:2)}\Big], \qquad \text{...(16)}$$

where $\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)$ and β are defined above. The conditional mean vector and conditional covariance matrix can be obtained from (15) and (16).

Multivariate dependence measures

The dependence among random variables is an important measure to study the relationship between the variables. In this subsection, we will obtain three important dependence measures for the multivariate transmuted family of distributions. These dependence measures include Kendall's multivariate coefficient of concordance, the multivariate local dependence function, and the multivariate version of Spearman's Rho. The results for the dependence measures are derived below.

The multivariate Kendall's Tau coefficient for joint continuous random variables, defined by Taylor (2016), is computed by using

$$\tau_p = \frac{2^p}{2^{p-1}-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_{\mathbf{x}}\left(\mathbf{x}\right)f_{\mathbf{x}}\left(\mathbf{x}\right)dx\ \cdots dx_p - \frac{1}{2^{p-1}-1}$$

$$= \frac{2^p}{2^{p-1}-1}I_K - \frac{1}{2^{p-1}-1}, \qquad \text{...(17)}$$

where

$$I_K = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_{\mathbf{x}}\left(\mathbf{x}\right)f_{\mathbf{x}}\left(\mathbf{x}\right)dx_1\cdots dx_p\ .$$

Now, using (5) and (6) in above equation, we have

$$I_K = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{i=1}^{p}G_i\left(x_i\right)\right\}\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-G_i\left(x_i\right)\right\}\right]$$

$$\times\left\{\prod_{i=1}^{p}g_i\left(x_i\right)\right\}\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_i\right)\right\}\right]dx_1\cdots dx_p.$$

Writing $\delta_i = \lambda_i + \lambda_{p+1}$, and making the transformation $G_i\left(x_i\right) = u_i$, we have

$$I_K = \int_0^1\cdots\int_0^1\left(\prod_{i=1}^{p}u_i\right)\left[1+\sum_{i=1}^{p}\delta_i\left(1-u_i\right)\right]$$

$$\left[1+\sum_{i=1}^{p}\delta_i\left(1-2u_i\right)\right]du_1\cdots du_p.$$

Solving the multiple integral, we have

$$I_K = \frac{1}{2^p}\left(1-\frac{2}{9}\sum_{i=1}^{p}\sum_{k=i+1}^{p}\delta_i\delta_k\right)$$

$$= \frac{1}{2^p}\left[1-\frac{2}{9}\sum_{i=1}^{p}\sum_{k=i+1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left(\lambda_k+\lambda_{p+1}\right)\right].$$

Using the value of I_K in (17) Kendall's multivariate coefficient of association for a multivariate transmuted family of distributions is

$$\tau_p = -\frac{2}{9\left(2^{p-1}-1\right)}\sum_{i=1}^{p}\sum_{k>i}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left(\lambda_k+\lambda_{p+1}\right). \quad ...(18)$$

It can be seen that, for $p = 2$, (18) reduces to the Kendall's coefficient of association for bivariate transmuted family of distribution given by Darwish *et al*. (2021).

Again, the Spearman Rho for a multivariate distribution is defined by Schmid and Schmidt (2007) as

$$\rho_p = \frac{2^p(p+1)}{2^p-(p+1)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_{\mathbf{x}}(\mathbf{x})\prod_{i=1}^{p}f_i(x_i)\,dx_1\cdots dx_p - \frac{(p+1)}{2^p-(p+1)}$$

$$= \frac{2^p(p+1)}{2^p-(p+1)}I_S - \frac{(p+1)}{2^p-(p+1)}, \quad ...(19)$$

where $I_S = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_{\mathbf{x}}(\mathbf{x})\prod_{i=1}^{p}f_i(x_i)\,dx_1\cdots dx_p$.

Now using the joint density function of multivariate transmuted family of distributions, from (5), and marginal density function of a single random variable, from (8), in above equation we have

$$I_S = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{i=1}^{p}G_i(x_i)\right\}\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-G_i(x_i)\right\}\right]\times\prod_{i=1}^{p}\left[g_i(x_i)\left\{1+\left(\lambda_i+\lambda_{p+1}\right)\left[1-2G_i(x_i)\right]\right\}\right]dx_1\cdots dx_p.$$

Writing $\delta_i = \lambda_i + \lambda_{p+1}$, and making the transformation $G_i(x_i) = u_i$, we have

$$I_S = \int_0^1\cdots\int_0^1\left(\prod_{i=1}^{p}u_i\right)\left[1+\sum_{i=1}^{p}\delta_i(1-u_i)\right]\left[\prod_{i=1}^{p}\left\{1+\delta_i(1-2u_i)\right\}\right]du_1\cdots du_p.$$

Solving the multiple integral, we have

$$I_S = \frac{1}{6^p}\left[\prod_{i=1}^{p}(3-\delta_i)+\sum_{i=1}^{p}\left\{\delta_i\prod_{\substack{j=1\\ j\neq i}}^{p}(3-\delta_j)\right\}\right]$$

$$= \frac{1}{6^p}\left[\prod_{i=1}^{p}\left(3-\lambda_i-\lambda_{p+1}\right)+\sum_{i=1}^{p}\left\{\left(\lambda_i+\lambda_{p+1}\right)\prod_{\substack{j=1\\ j\neq i}}^{p}\left(3-\lambda_i-\lambda_{p+1}\right)\right\}\right].$$

Using this value in (19), we have

$$\rho_p = \frac{(p+1)}{3^p\left[2^p-(p+1)\right]}\left[\prod_{i=1}^{p}\left(3-\lambda_i-\lambda_{p+1}\right)\left\{1+\sum_{i=1}^{p}\frac{\left(\lambda_i+\lambda_{p+1}\right)}{\left(3-\lambda_i-\lambda_{p+1}\right)}\right\}-3^p\right]. \quad ...(20)$$

It is easy to see that, for $p = 2$, (20) reduces to the Spearman's coefficient of association for a bivariate transmuted family of distribution given by Darwish *et al*. (2021).

A local dependence function for two absolutely continuous random variables has been defined by Holland and Wang (1987) as

$$\gamma(x_1,x_2) = \frac{\partial^2}{\partial x_1\partial x_2}\ln f(x_1,x_2).$$

The multivariate extension the above dependence function is immediately written as

$$\gamma\left(x_1,x_2,\ldots,x_p\right) = \frac{\partial^p}{\partial x_1\,\partial x_2\cdots\partial x_p}\ln f\left(x_1,x_2,\ldots,x_p\right),$$

where $f\left(x_1,x_2,\ldots,x_p\right)$ is the multivariate density function. The multivariate local dependence function for the multivariate transmuted family of distribution is

$$\gamma\left(x_1,x_2,\ldots,x_p\right) = -\frac{2^p(p-1)!\prod_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)g_i(x_i)}{\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_p\right)\left\{1-2G_i(x_i)\right\}\right]^p}. \quad ...(21)$$

We can see that Kendall's coefficient of association and Spearman's Rho remains the same irrespective of the baseline distribution but the multivariate local dependence function involves the density and distribution function of the baseline distribution.

Estimation of parameters

In the following, we will discuss the maximum likelihood estimation of the parameters of multivariate transmuted family of distributions assuming that all the parameters of $G_i\left(x_i\right)$ are known for $i=1,2,\ldots,p$. First, suppose that $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$ is a random sample of *n* vector observations from the multivariate transmuted density function, given in (6). The likelihood function is therefore

$$L=\prod_{j=1}^{n}\left\{\prod_{i=1}^{p}g_i\left(x_{ij}\right)\right\}\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_{ij}\right)\right\}\right].$$

The log–likelihood function is

$$\ell=\sum_{j=1}^{n}\sum_{i=1}^{p}\ln\left[g_i\left(x_{ij}\right)\right]+\sum_{j=1}^{n}\ln\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_{ij}\right)\right\}\right]. \quad ...(22)$$

The derivatives of log–likelihood function with respect to λ_i ; $i=1,2,\ldots,p+1$ are

$$\frac{\partial\ell}{\partial\lambda_i}=\sum_{j=1}^{n}\frac{1-2G_i\left(x_{ij}\right)}{\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_{ij}\right)\right\}\right]}\ ;\ i=1,2,\ldots,p$$

and
$$\frac{\partial\ell}{\partial\lambda_{p+1}}=\sum_{j=1}^{n}\frac{2\left[1-\sum_{i=1}^{p}G_i\left(x_{ij}\right)\right]}{\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{1-2G_i\left(x_{ij}\right)\right\}\right]}.$$

The maximum likelihood estimators of λ_i ; $i=1,2,\ldots,p+1$ can be obtained by equating the above derivatives to zero and numerically solving the resulting equations.

The proposed multivariate transmuted family of distributions can be explored for any baseline distributions. In the following section, we will study the multivariate transmuted family of distributions for the baseline Weibull distribution and the resulting distribution is named as multivariate transmuted Weibull distribution.

The multivariate transmuted Weibull distribution

In this sub-section, we have proposed the multivariate transmuted Weibull (MTW) distribution by using the following density and distribution functions of the Weibull distribution

$$g_i\left(x_i\right)=\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}} \text{ and } G_i\left(x_i\right)=1-\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\ ;\quad x_i,\theta_i,\alpha_i>0 \quad ...(23)$$

in (5). The *cdf* of the proposed *MTW* distribution is

$$F_{\mathbf{x}}\left(\mathbf{x}\right)=\left[\prod_{i=1}^{p}\left\{1-\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right\}\right]\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right];\ \mathrm{x},\ \alpha,\ \theta>0, \quad ...(24)$$

where

$\mathbf{x}=\begin{bmatrix}X_1 & X_2 & \cdots & X_p\end{bmatrix}'$, $\boldsymbol{\alpha}=\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p\end{bmatrix}'$, $\boldsymbol{\theta}=\begin{bmatrix}\theta_1 & \theta_2 & \cdots & \theta_p\end{bmatrix}'$ and λ_i are transmutation parameters such that $\left(\lambda_i+\lambda_{p+1}\right)\in\left[-1,1\right]$ for $i=1,2,\ldots,p$ and $-1\le\sum_{i=1}^{p}\lambda_i+p\lambda_{p+1}\le1$. The density function corresponding to (24) is

$$f_{\mathbf{x}}\left(\mathbf{x}\right)=\left[\prod_{i=1}^{p}\left\{\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right\}\right]\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right];\ \mathrm{x},\ \alpha,\ \theta>0. \quad ...(25)$$

Some properties of the proposed *MTW* distribution will be studied in the following section.

Properties of multivariate transmuted Weibull distribution

In this sub-section, we will discuss some important properties of the proposed *MTW* distribution. These properties include marginal and conditional distributions and the conditional moments.

The marginal distributions

We have seen above that the marginal distribution of any random variable of multivariate transmuted family of distributions is a transmuted distribution. We have also seen that the joint marginal distribution of a pair of random variables is a bivariate transmuted distribution, proposed by Darwish *et al*. (2021). Using these results we can readily say that the marginal distribution of any variable X_i in an *MTW* distribution is a transmuted Weibull distribution, proposed by Khan *et al*. (2017), with density

$$f_{X_i}\left(x_i\right)=\left[\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right]\left[1+\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right];\ x_i,\theta_i,\alpha_i>0.$$

The joint marginal distribution of any pair of random variables in an *MTW* distribution is a bivariate transmuted Weibull distribution, proposed by Darwish *et al*. (2021), with joint density function

$$f_{X_i,X_m}\left(x_i,x_m\right)=\left[\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right]\left[\frac{\alpha_m}{\theta_m^{\alpha_m}}x_m^{\alpha_m-1}\mathrm{e}^{-\left(x_m/\theta_m\right)^{\alpha_m}}\right]\times\left[1+\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}+\left(\lambda_m+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_m/\theta_m\right)^{\alpha_m}}-1\right\}\right];\ x_i,x_m>0.$$

The marginal distribution of any subset of variables, say $_1=\left[X_1\quad X_2\quad\cdots\quad X_t\right]'$, in an *MTW* distribution is again *MTW* with density and distribution functions given as

$$f_{\mathbf{x}_1}\left(\mathbf{x}_1\right)=\left[\prod_{i=1}^{t}\left\{\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right\}\right]\left[1+\sum_{i=1}^{t}\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right];\ \mathrm{x}_1,\ \alpha,\ \theta>0, \quad ...(26)$$

and

$$F_{\mathbf{x}_1}\left(\mathbf{x}_1\right)=\left[\prod_{i=1}^{t}\left\{1-\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right\}\right]\left[1+\sum_{i=1}^{t}\left(\lambda_i+\lambda_{p+1}\right)\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right];\ \mathrm{x}_1,\ \alpha,\ \theta>0. \quad ...(27)$$

The conditional distributions

The conditional distributions in the case of a multivariate transmuted family of distributions are discussed above. In the following, we will discuss the conditional distributions for the *MTW* distribution.

The conditional distribution of any variable X_i given the information of other variables for a multivariate transmuted family of distributions is given in (12). Now, using the density and distribution functions of the Weibull distribution, the conditional density function of X_i given the other variables is

$$f\left(x_i\mid\mathbf{x}_{(i)}\right)=\frac{1}{\Delta_i\left(\mathbf{x}_{(i)}\right)}\left[\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right],\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right];\ x_i>0 \quad ...(28)$$

where

$$\Delta_i\left(\mathbf{x}_{(i)}\right)=\left[1+\sum_{j\neq i=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right].$$

The joint conditional distribution of any pair of random variables given the information of other variables for a multivariate transmuted family of distributions is given in (13) which, for the case of the *MTW* distribution, is

$$f\left(x_i,\mathrm{x}_m\mid\mathbf{x}_{(i,m)}\right)=\frac{1}{\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)}\left[\frac{\alpha_i}{\theta_i^{\alpha_i}}x_i^{\alpha_i-1}\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right]\left[\frac{\alpha_m}{\theta_m^{\alpha_m}}x_m^{\alpha_m-1}\mathrm{e}^{-\left(x_m/\theta_m\right)^{\alpha_m}}\right]\times\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right];\ \left(x_i,x_m\right)>0, \quad ...(29)$$

where

$$\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)=\left[1+\sum_{j\neq(i,m)=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right].$$

The conditional distribution of one subset of variables given the information of other subset of variables in the case of *MTW* distribution can be obtained by using the density and distribution functions of Weibull random variables in (14) and is given as

$$f\left(\mathbf{x}_1 \mid \mathbf{x}_2\right)=\frac{1}{\Delta_t\left(\mathbf{x}_2\right)}\left[\prod_{i=1}^{t}\left\{\frac{\alpha_i}{\theta_i^{\alpha_i}} x_i^{\alpha_i-1} \mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}\right\}\right],$$

$$\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right];\ \mathrm{x}_1>0$$

where

$$\Delta_t\left(\mathbf{x}_2\right)=\left[1+\sum_{j=t+1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right].$$

In the following subsection we will discuss the marginal and conditional moments for the *MTW* distribution.

The marginal and conditional moments

The moments of a distribution are useful in studying the properties of the distribution. The marginal and joint marginal moments in case of the *MTW* distribution can be easily obtained by using the marginal distribution of a single random variable and the joint marginal distribution of two random variables. The joint moments in case of two random variables are discussed by Darwish *et al*. (2020). In the following, we will obtain the single and joint conditional moments for the parent *MTW* distribution.

The *r*th conditional moment of a single random variable given the information of other random variables is given in (15). We can see that the conditional moment for a multivariate transmuted family of distributions is based upon the raw moments of the parent distribution and the raw moments of the larger observation in a sample of size 2 from the parent distribution $G_i\left(x_i\right)$. Now to compute the conditional moment for the *MTW* distribution we first see that the *r*th raw moment and *r*th moment of the larger observation in a sample of size 2 from the Weibull distribution, given in (23), are

$$\mu_X^{r}=\theta^r\Gamma\left(\frac{r}{\alpha}+1\right) \text{ and } \mu_{X(2:2)}^{r}=2\theta^r\Gamma\left(\frac{r}{\alpha}+1\right)\left(1-2^{-(r/\alpha+1)}\right).$$

Now, using these in (15), the *r*th conditional moment of X_i given the information of other random variables for *MTW* distribution is

$$\mu_{X_i|\mathbf{x}_{(i)}}^{r}=\frac{\theta_i^{\alpha_i}}{\Delta_i\left(\mathbf{x}_{(i)}\right)}\Gamma\left(\frac{r}{\alpha_i}+1\right)\left[\beta-2\sum_{j\neq i=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\right.$$

$$\left.\left\{1-\mathrm{e}^{-\left(x_j/\theta_j\right)^{\alpha_j}}\right\}-2\left(\lambda_i+\lambda_{p+1}\right)\left(1-2^{-(r/\alpha_i+1)}\right)\right] \quad \text{...(30)}$$

where

$$\beta=1+\sum_{i=1}^{p}\lambda_i+p\lambda_{p+1} \text{ and}$$

$$\Delta_i\left(\mathbf{x}_{(i)}\right)=\left[1+\sum_{j\neq i=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{2\mathrm{e}^{-\left(x_i/\theta_i\right)^{\alpha_i}}-1\right\}\right].$$

The joint conditional moment for two random variables given the information of other random variables for a multivariate transmuted family of distributions is given in (16). Now, the same for *MTW* distribution is given as

$$\mu_{X_i,X_m|\mathbf{x}_{(i,m)}}^{r}=\frac{1}{\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)}\left[\beta\mu_{x_i}^{r}\mu_{x_m}^{s}-2\mu_{x_i}^{r}\mu_{x_m}^{s}\sum_{j\neq(i,m)=1}^{p}\right.$$

$$\left(\lambda_j+\lambda_{p+1}\right)G_j\left(x_j\right)-\left(\lambda_i+\lambda_{p+1}\right)$$

$$\left.\mu_{x_i(2:2)}^{r}\mu_{x_m}^{s}-\left(\lambda_k+\lambda_{p+1}\right)\mu_{x_i}^{r}\mu_{x_m(2:2)}^{s}\right],$$

$$\mu_{X_i,X_m|\mathbf{x}_{(i,m)}}^{r,s}=\frac{\theta_i^r\theta_m^s}{\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)}\Gamma\left(\frac{r}{\alpha_i}+1\right)\Gamma\left(\frac{s}{\alpha_m}+1\right)$$

$$\left[\beta-2\sum_{j\neq(i,m)=1}^{p}\left(\lambda_j+\lambda_{p+1}\right)\left\{1-\mathrm{e}^{-\left(x_j/\theta_j\right)^{\alpha_j}}\right\}\right.$$

$$-2\left(\lambda_i+\lambda_{p+1}\right)\left(1-2^{-(r/\alpha_i+1)}\right)$$

$$\left.-2\left(\lambda_m+\lambda_{p+1}\right)\left(1-2^{-(r/\alpha_m+1)}\right)\right], \quad \text{...(31)}$$

where β and $\Delta_{i,m}\left(\mathbf{x}_{(i,m)}\right)$ are defined above. The conditional moments are useful in computing conditional means, conditional variances and conditional covariances for *MTW* distribution.

Parameter estimation for multivariate transmuted Weibull distribution

In this sub-section, we will discuss the maximum likelihood estimation for parameters of the *MTW* distribution. For this suppose $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$ is a random sample of n vector observations from the *MTW* distribution. The likelihood function for the given sample is

$$L=\prod_{j=1}^{n}\left[\left\{\prod_{i=1}^{p}\left(\frac{\alpha_i}{\theta_i^{\alpha_i}}x_{ij}^{\alpha_i-1}\mathrm{e}^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}\right)\right\}\right.$$

$$\left.\left\{1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left(2\mathrm{e}^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right)\right\}\right],$$

which has $(3p+1)$ unknown parameters. The log-likelihood function is

$$\ell = \sum_{i=1}^{p}\left[n\ln\alpha_i - n\alpha_i \ln\theta_i + (\alpha_i - 1)\sum_{j=1}^{n}\ln x_{ij} - \sum_{j=1}^{n}\left(x_{ij}/\theta_i\right)^{\alpha_i}\right] + \sum_{j=1}^{n}\ln\left[1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right\}\right]. \quad ...(32)$$

The derivatives of the log-likelihood function with respect to unknown parameters are

$$\frac{\partial\ell}{\partial\theta_i} = -\frac{n\alpha_i}{\theta_i} + \frac{\alpha_i}{\theta_i^2}\sum_{j=1}^{n}\left\{x_{ij}\left(x_{ij}/\theta_i\right)^{\alpha_i-1}\right\} + \frac{2\alpha_i}{\theta_i^2}\left(\lambda_i+\lambda_{p+1}\right)\sum_{j=1}^{n}\frac{x_{ij}\left(x_{ij}/\theta_i\right)^{\alpha_i-1}e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}}{1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right\}};$$

$$i = 1,2,\ldots,p,$$

$$\frac{\partial\ell}{\partial\alpha_i} = -\frac{n}{\alpha_i} - n\ln\theta_i + \sum_{j=1}^{n}\ln x_{ij} - \sum_{j=1}^{n}\left\{\left(x_{ij}/\theta_i\right)^{\alpha_i}\ln\left(x_{ij}/\theta_i\right)\right\} - 2\left(\lambda_i+\lambda_{p+1}\right)\sum_{j=1}^{n}\frac{\left(x_{ij}/\theta_i\right)^{\alpha_i}\ln\left(x_{ij}/\theta_i\right)e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}}{1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right\}};$$

$$i = 1,2,\ldots,p,$$

$$\frac{\partial\ell}{\partial\lambda_i} = \sum_{j=1}^{n}\frac{2e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1}{1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right\}};$$

$$i = 1,2,\ldots,p,$$

and

$$\frac{\partial\ell}{\partial\lambda_{p+1}} = \sum_{j=1}^{n}\frac{2\left(\sum_{i=1}^{p}e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right)}{1+\sum_{i=1}^{p}\left(\lambda_i+\lambda_{p+1}\right)\left\{2e^{-\left(x_{ij}/\theta_i\right)^{\alpha_i}}-1\right\}}.$$

The maximum likelihood estimators of unknown parameters are obtained by equating the above $(3p+1)$ derivatives to zero and numerically solving the resulting $(3p+1)$ equations in $(3p+1)$ unknowns.

Real data applications

In this sub-section, real data applications of the *MTW* distribution are given. The real data applications are done by using daily *COVID–19* data of the World for three recent dates; November 20–22, 2020. The data can be accessed at *www.worldometer.com*. Since the data on daily *COVID–19* cases represents counts, we have used the logarithmic transformation of the data to make it continuous. We have carried out the data analysis on overall data and data split with respect to the median of the log of daily cases. We have divided the transformed daily *COVID–19* cases of the World into two groups; one below the median of the log of daily cases and one above. The summary statistics for the overall data and two groups are given in Table 1.

The correlation coefficients between different variables for the whole data and the two subsets are given in Table 2.

From Table 2 we can see that the variables have high pairwise correlation. Also all the correlations are significant at 1%. Infact the correlation coefficients are significant at 0.1%. These correlation coefficients indicate that the variables are jointly dependent upon each other and hence they should be modelled by using some trivariate distribution.

We have modelled these three data sets by using three trivariate distributions. The distributions that we have fitted include the trivariate transmuted Weibull (*TrTW*) distribution, obtained by using *p* = 3 in (6), the FGM trivariate Weibull (*FGMTW*) distribution obtained by using the Weibull distribution in Gumble copula and a trivariate the Weibull distribution, the HS trivariate Weibull (*HSTW*) distribution, introduced by Hanif Shahbaz *et al*. (2012).

The density functions of *FGMTW* and *HSTW* distributions are, respectively

Table 1: Summary statistics of the data

Data	Date	n	Min	Mean	Q_1	Median	Q_3	Skew	Max
Whole	20-11(X_1)	144	0.000	6.021	4.100	6.276	7.963	-0.242	12.186
	21-11(X_2)	144	0.000	6.086	4.131	6.470	7.864	-0.168	12.090
	22-11(X_3)	144	0.000	5.918	3.965	6.163	7.643	-0.309	11.869
Below Median	20-11(Y_1)	72	0.000	3.907	2.901	4.057	5.443	-0.448	6.219
	21-11(Y_2)	72	0.000	4.077	3.032	4.119	5.421	-0.319	6.465
	22-11(Y_3)	72	0.000	3.883	2.785	3.940	5.324	-0.607	6.071
Above Median	20-11(Z_1)	72	6.333	8.134	7.231	7.972	8.771	0.766	12.186
	21-11(Z_2)	72	6.475	8.096	7.158	7.869	8.781	0.894	12.090
	22-11(Z_3)	72	6.256	7.952	7.021	7.655	8.583	0.798	11.869

Table 2: Correlation coefficient between various variables

Whole data			Below median			Above median		
$r_{X_1X_2}$	$r_{X_1X_3}$	$r_{X_2X_3}$	$r_{Y_1Y_2}$	$r_{Y_1Y_3}$	$r_{Y_2Y_3}$	$r_{Z_1Z_2}$	$r_{Z_1Z_3}$	$r_{Z_2Z_3}$
0.9845 p<0.01	0.9763 p<0.01	0.9834 p<0.01	0.7603 p<0.01	0.9173 p<0.01	0.7377 p<0.01	0.6995 p<0.01	0.9514 p<0.01	0.6605 p<0.01

$$f_{FGMTW}(x_1,x_2,x_3)=\frac{\alpha_1\alpha_2\alpha_3x_1^{\alpha_1-1}x_2^{\alpha_2-1}x_3^{\alpha_3-1}}{\theta_1^{\alpha_1}\theta_2^{\alpha_2}\theta_3^{\alpha_3}}\left[1+\lambda_1\left\{1-2e^{-(x_1/\theta_1)^{\alpha_1}}\right\}\left\{1-2e^{-(x_2/\theta_2)^{\alpha_2}}\right\}\left\{1-2e^{-(x_3/\theta_3)^{\alpha_3}}\right\}\right] \quad ...(33)$$

and

$$f_{HSTW}(x_1,x_2,x_3)=\frac{\alpha_1\alpha_2\alpha_3x_1^{3\alpha_1-1}x_2^{2\alpha_2-1}x_3^{\alpha_3-1}}{\theta_1^{3\alpha_1}\theta_2^{2\alpha_2}\theta_3^{\alpha_1}}\exp\left[-\left(\frac{x_1}{\theta_1}\right)^{\alpha_1}\left\{1+\left(\frac{x_2}{\theta_2}\right)^{\alpha_2}+\left(\frac{x_2}{\theta_2}\right)^{\alpha_2}\left(\frac{x_3}{\theta_3}\right)^{\alpha_3}\right\}\right], \quad ...(34)$$

where $(x_1,x_2,x_3)>0$ and all the parameters are positive for both of the distributions, except $\lambda_1 \in [-1,1]$.

Table 3: Fitted distributions for the whole COVID–19 data

Parameters	Distributions		
	TrTW	FGMTW	HSTW
θ_1	7.1940	0.0001	6.0392
θ_2	6.0676	4.12 x 10^{-7}	7.1671
θ_3	7.0621	0.0002	7.9643
α_1	2.7534	0.4576	0.9725
α_2	2.4721	0.4584	1.3277
α_3	2.7776	0.4572	2.1416
λ_1	0.3773	0.1171	
λ_2	-0.6227		
λ_3	0.3773		
λ_4	-0.0705		
Log-likelihood	-986.8249	-1814.5921	-1288.9752
AIC	1993.6498	3643.1842	2589.9504
BIC	2023.3479	3663.9729	2607.7693

We have fitted these distributions on three data sets by computing the maximum likelihood estimates of the unknown parameters. The maximum likelihood estimates of unknown parameters are computed by using the *R* package "maxLik" introduced by Henningsen & Toomet (2011). The performance of the distributions are assessed by computing the Akaike's information criterion (*AIC*) and the Bayesian information criterion (*BIC*). The results of these analyses are given in Tables 3–5. Table 3 contains results for the whole data, Table 4 has results for data below median cases and Table 5 contains results for data above median cases.

Table 4: Fitted distributions for the daily COVID–19 data below the median

Parameters	Distributions		
	TrTW	FGMTW	HSTW
θ_1	4.6627	0.0006	12.1665
θ_2	4.0476	1.53 x 10^{-7}	20.5443
θ_3	4.6437	0.0006	5.6202
α_1	2.9923	0.4432	14.5263
α_2	2.6842	0.4448	18.7934
α_3	2.9506	0.4428	19.4948
λ_1	0.3654	0.1144	
λ_2	–0.6344		
λ_3	0.3655		
λ_4	–0.1069		
Log-likelihood	–391.6393	–1651.8421	–790.0924
AIC	803.2786	3317.6842	1592.1848
BIC	826.0453	3333.6209	1605.8448

Table 5: Fitted distributions for the daily COVID–19 data above the median

Parameters	Distributions		
	TrTW	FGMTW	HSTW
θ_1	8.4667	3.55 x 10^{-7}	8.1952
θ_2	9.1262	7.77 x 10^{-7}	7.2235
θ_3	8.2789	0.0002	9.2567
α_1	6.1104	0.4633	12.2257
α_2	7.3003	0.4632	10.5287
α_3	6.0972	0.4627	11.6772
λ_1	-0.3147	0.1176	
λ_2	0.6852		
λ_3	-0.3147		
λ_4	0.0464		
Log-likelihood	-359.3084	-1584.8859	-663.42
AIC	738.6168	3183.7718	1338.8400
BIC	761.3835	3199.7085	1352.5000

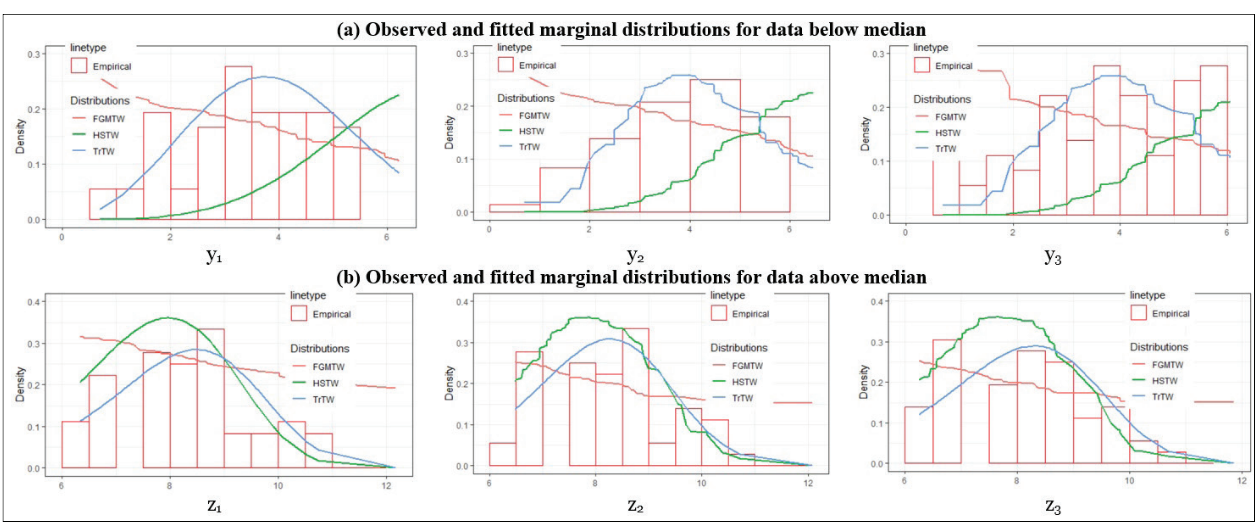

Figure 1: Observed and fitted marginal distributions for data below median (a) and above median (b)

It can be easily seen that the *AIC* and *BIC* values of the proposed trivariate transmuted Weibull distributions for the whole data and two sub data sets are smallest indicating that the trivariate transmuted Weibull distribution is the best fit for logarithems of the daily COVID–19 cases.

The plot of observed and fitted marginal distributions for data below median (Panel–a) and above median (Panel–b) are given in Figure 1, below. The plots of observed data and fitted marginal distributions also indicates that the trivariate transmuted Weibull distribution is the most suitable distribution for capturing behaviour of the data. Also, this distribution is the most appropriate for capturing tail behaviour.

CONCLUSION

In this paper, we have proposed a new multivariate transmuted family of distributions. Some properties of the proposed family have been studied. We have also obtained expressions for the maximum likelihood estimation of the parameters of the family of distributions. The proposed multivariate family provides the transmuted family of distributions, proposed by Shaw and Buckley (2007), and the bivariate transmuted family of distributions, proposed by Darwish *et al*. (2020), as special case. The proposed multivariate family has been studied for parent Weibull distribution giving rise to the multivariate transmuted Weibull (*MTW*) distribution. Some properties of the proposed *MTW* distribution alongside the maximum likelihood estimation of the parameters has been discussed. We have applied the *MTW* distribution on daily COVID–19 cases of the world. We have seen that the proposed *MTW* turned out to be the best fit for modeling of the data used. The multivariate transmuted family of distributions can be further explored for any baseline distributions.

Conflicts of interest

The authors declare no conflict of interest regarding the publication of this paper.

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