J.Natn.Sci.Foundation Sri Lanka 2019 47 (1): 105 - 112
DOI: http://dx.doi.org/10.4038/jnsfsr.v47i1.8932

RESEARCH ARTICLE

$\mathbb{Z}_p$ as a final coalgebra obtained by Cauchy completing the initial algebra

Annanthakrishna Manokaran[1], Jayampathy Ratnayake[2*] and Romaine Jayewardene[2]
[1] *Department of Mathematics and Statistics, Faculty of Science, University of Jaffna, Jafna.*
[2] *Department of Mathematics, Faculty of Science, University of Colombo, Colombo 03.*

Submitted: 02 April 2018; Revised: 29 August 2018; Accepted: 26 October 2018

Abstract: The metric space of *p*-adic integers with the *p*-adic metric, $\mathbb{Z}_p$, is presented as a final coalgebra, obtained as the Cauchy completion of the initial algebra of an endofunctor on the category of one-pointed one-bounded metric spaces with short maps. This fact, that the final coalgebra is the Cauchy completion of the initial algebra, is used to show that $\mathbb{Z}_p$ is also the final coalgebra of this endofunctor in the continuous setting.

Some final coalgebras on pointed metric spaces with short maps are known to be the Cauchy completion of the intial algebra. In a separate study, $\mathbb{Z}_p$ has been observed as the final coalgebra of certain endofunctors on ultra metric spaces. The results of this paper unify these observations and give a coalgebraic characterisation of the self similarity of $\mathbb{Z}_p$, while relaxing the ultra metric condition to one-bounded metrics. Obtaining the final coalgebra as the Cauchy completion of the initial algebra is in close analogy with classical results in iterated function systems. Another question that has been asked in the literature is whether such results hold when the maps are chosen to be Lipchitz. We give evidence as to why selecting continuous maps as the morphisms may be the right choice for such results to hold.

Keywords: Cauchy completion, final coalgebra, initial algebra, *p*-adic integers.

INTRODUCTION

It has been shown in Bhattacharya *et al*. (2014) and Moss *et al*. (2013) that in certain cases, the final coalgebra of an endofunctor defined on a category of metric spaces can be obtained as the Cauchy completion of the initial algebra. Specifically, Bhattacharya *et al*. (2014) and Moss *et al*. (2013) considered some endofunctors on the category of *i*-pointed metric spaces (where $i = 2,3$) with short maps preserving distinguished elements as the choice of morphisms and showed that the final coalgebra of these endofunctors is the Cauchy completion of the initial algebra.

In a separate study, Bhattacharya (2015) has demonstrated $\mathbb{Z}_p$, the set of *p*-adic integers (where *p* is a prime), as the final coalgebra of an endofunctor on the category of one-bounded ultra metric spaces with short maps. Bhattacharya (2015) also exhibits $\mathbb{N} \cup \{0\} = \mathbb{N}^*$ equipped with the *p*-adic metric as the initial algebra of a similar but different endofunctor on the category of one-pointed one-bounded ultra metric spaces with short maps.

The main motivation of this paper is to unify the above observations and strengthen the postulate that under suitable conditions '*the final coalgebra is the Cauchy completion of the initial algebra*'. This paper demonstrates $\mathbb{Z}_p$ as the final coalgebra of an endofunctor on the category of one-pointed metric spaces with short maps and most importantly, shows that it can be obtained as the Cauchy completion of the initial algebra of the same endofunctor. Finally, these observations are applied to obtain the final coalgebra of the same functor with the choice of morphisms as continuous maps. Results at the beginning of this paper are motivated by Bhattacharya

* Corresponding author (jratnaya@maths.cmb.ac.lk; https://orcid.org/0000-0001-5191-3895)

et al. (2014); however, the discussion on continuous maps is new and applicable to the cases considered in that paper. Comparing with Bhattacharya (2015), this work emphasises on the one-pointed case, relaxes the ultra metric condition, works with an approach which allows us to deal with both short maps and continuous maps, and identifies the main objects of interest as the initial algebra and final coalgebra of the same functor where the latter is the Cauchy completion of the former.

We will use the following terminologies and notations throughout this paper. A one-pointed set is a set having one distinguished element. Such a set is denoted by a pair $(X,*)$, where $* \in X$ is the distinguished element of X. We will often omit the distinguished point from the description of the set and simply write, 'Let X be a one-pointed set'. To differentiate the distinguished points of two (or more) one-pointed sets X and Y, we will use subscripts and write $*_X$ and $*_Y$ for $*$ of X and $*$ of Y, respectively. There is the category of one-pointed sets, denoted by Set_1, whose objects are one-pointed sets and morphisms are functions preserving the distinguished elements. A one-pointed metric space (X,d) is a one -pointed set $X = (X,*)$ equipped with a one-bounded metric d; i.e., $d(x,y) \leq 1, \forall x,y \in X$.

The class of one-pointed metric spaces can be raised to the categories Met_1^S, Met_1^L and Met_1^C, where the choice of morphisms are respectively short maps, Lipschitz maps and Continuous maps that preserve the distinguished element. The superscript denotes the choice of the morphisms and the subscript 1 emphasises the one-pointed case. Note that Met_1^L and Met_1^S are subcategories of Met_1^C, Met_1^S is a subcategory of Met_1^L and there is a forgetful functor from these categories (Met_1^S, Met_1^L and Met_1^C) to Set_1.

One-pointed metric spaces and the endofunctor F_1 on Set_1 defined as follows are central to the discussion of this paper. Fix a prime number p and let $V_p = \{0, 1, \cdots, p-1\}$. For a one-pointed set $(X,*)$, we define $F_1X = X \times V_p$, which is a one-pointed set with the distinguished element $(*,0)$. Given a morphism $f : X \to Y$ of Set_1, $F_1 f : X \times V_p \to Y \times V_p$ is given by $F_1 f(x,a) = (f(x), a)$. $F_1 f$ preserves the distinguished point as $F_1 f(*_X, 0) = (f(*_X), 0) = (*_Y, 0)$. Thus, this definition makes F_1 an endofunctor on Set_1.

The endofunctor F_1 can be lifted to an endofunctor on Met_1^C as follows. Given a one-pointed metric space (X,d), $\frac{1}{p}X$ represents the metric space obtained by contracting X by a factor of $\dfrac{1}{p}$; i.e., $d_{\frac{1}{p}X}(x,y) = \frac{1}{p}d_X(x,y)$. The set V_p is given the discrete metric. We let $F_1X = \frac{1}{p}X \times V_p$ and give it the maximum metric; namely, $d\left((x_1,a_1),(x_2,a_2)\right) = \max\{\frac{1}{p}d(x_1,x_2), d(a_1,a_2)\}$. Here, we have used the same letter d to identify the metric on $\frac{1}{p}X$, on V_p and on $\frac{1}{p}X \times V_p$, since it is understood from the context to which metric we refer. Clearly, the metric on F_1X is one-bounded.

As in the set case, let the distinguished element of $\frac{1}{p}X \times V_p$ be $(*_X, 0)$. Thus, $F_1X = \frac{1}{p}X \times V_p$ is a one-pointed metric space. Given a continuous function f on a one-pointed metric space, $F_1 f$ as in the set setting is the application of f to the first coordinate. Part (i) of Lemma 1 guarantees that F_1 is an endofunctor on Met_1^C.

Lemma 1. Let X and Y be two one-pointed metric spaces. If f : $f : X \to Y$ has any of the following properties, then so does $F_1 f$.

i). Continuous

ii). Lipschitz

iii). Short map

Proof. The proof is straightforward and therefore we will outline only the Lipschitz case. The short map setting follows from the first case of the Lipschitz setting. Let f be a Lipschitz continuous map with the Lipschitz constant k.

When $k \leq 1$, we have $d_Y(f(x_1), f(x_2)) \leq k d_X(x_1,x_2) \leq d_X(x_1,x_2)$. Thus,

$$\begin{aligned} d_{\frac{1}{p}Y\times V_p}(F_1 f(x_1,a_1), F_1 f(x_2,a_2)) &= \\ &\max\{\frac{1}{p}d_Y(f(x_1),f(x_2)), d_{V_p}(a_1,a_2)\} \\ &\leq \max\{\frac{1}{p}d_X(x_1,x_2), d_{V_p}(a_1,a_2)\} \\ &= d_{\frac{1}{p}X\times V_p}((x_1,a_1),(x_2,a_2)), \end{aligned}$$

as required.

When $k > 1$ we have,

$$\begin{aligned} d_{\frac{1}{p}Y\times V_p}(F_1 f(x_1,a_1), F_1 f(x_2,a_2)) &= \\ &\max\{\frac{1}{p}d_Y(f(x_1),f(x_2)), d_{V_p}(a_1,a_2)\} \\ &\leq \max\{\frac{k}{p}d_X(x_1,x_2), d_{V_p}(a_1,a_2)\} \\ &\leq k\max\{\frac{1}{p}d_X(x_1,x_2), d_{V_p}(a_1,a_2)\} \\ &= k\, d_{\frac{1}{p}X\times V_p}((x_1,a_1),(x_2,a_2)). \end{aligned}$$

In both cases $F_1 f$ satisfies the Lipschitz condition; i.e., $F_1 f$ has the Lipschitz constant 1 or k provided that f has the Lipschitz constant $k \leq 1$ or $k > 1$, respectively. This completes the proof. □

Parts (ii) and (iii) of Lemma 1 guarantee that F_1 on Met_1^C restricts to an endofunctor on the categories Met_1^L and Met_1^S, respectively.

Before we discuss the final coalgebra of F_1, let us recall some basic facts about p-adic integers [see also (Cuoco, 1991) and (Serre, 2012)] and introduce some notational conventions used in this paper. The set of p-adic integers (p is a prime) can be identified with the set of power series of the form $\sum_0^\infty a_i p^i$, where each $a_i \in V_p$. In this paper, such a power series (i.e., a p-adic integer) will be denoted by the infinite stream $a = (\cdots, a_2, a_1, a_0)$. The set of p-adic integers forms a metric space with respect to the p-adic metric d_p. The distance between two p-adic numbers a and b is given by $d_p(a,b) = p^{-i}$, where i is the largest integer $i \geq 0$ such that $p^i \mid a-b$; i.e., $d_p(a,b) = p^{-i}$ if $a_k = b_k$ for $k \leq i-1$, and $a_i \neq b_i$. It also follows directly from the definition that d_p is a one-bounded metric. The one-bounded metric space of all p-adic integers is denoted by $\mathbb{Z}_p$. By choosing $0 = (\cdots, 0, 0)$ as the distinguished point, we lift $\mathbb{Z}_p$ to a one-pointed (and one-bounded) metric space.

The next section of this paper investigates the final coalgebra and the initial algebra of the endofunctor F_1 on Met_1^S and shows that the final coalgebra can be obtained by Cauchy completing the initial algebra. In a latter section, this fact is used to compute the mediating morphism from a coalgebra to the final coalgebra in the continuous setting (Proposition 1) and its implications are stated. It is shown there that the final coalgebra in the continuous setting is the same as the final coalgebra in the short map setting. Finally, a brief look into the initial algebra in the continuous and Lipschitz settings is considered, motivated by the questions 'Is the initial algebra, if it exists, in the continuous and Lipschitz settings the same as the initial algebra in the short map setting?' and 'If they exist, can we extend the obtained results to these cases?' [see (Bhattacharya *et al*., 2014) and (Moss *et al*., 2013)].

The presentation of $\mathbb{Z}_p$ as a final coalgebra gives a characterisation of its self-similar nature as in Leinster, (2011) [see also (Cuoco, 1991)] and observing it as the Cauchy completion of the initial algebra is in close analogy to results in Hutchinson (1981) [see also (Bhattacharya *et al*., 2014)].

INITIAL ALGEBRA AND FINAL COALGEBRA OF F_1 ON Met_1^S

The initial algebra of F_1 on Met_1^S can be found by direct application of Adámek's theorem [see (Adámek, 1974) or Theorem 3.17 of Adámek *et al*. (2010)]. The initial object of Met_1^S is the singleton set. The colimit of the initial chain starting from the initial object exists and is preserved by the functor F_1. It can be easily shown that the colimit of the initial chain in the associated category of metric spaces is $(\mathbb{N}^*, d_p)$. The structure map ϕ of the initial algebra is constructed in the natural way (as given in Lemma 2). The proofs of these facts, summarised in the following Lemma, are straightforward and the details are omitted.

Lemma 2. The initial algebra of F_1 on Met_1^S is the pair $((\mathbb{N}^*, d_p), \phi)$, where $\phi : F_1\mathbb{N}^* \longrightarrow \mathbb{N}^*$ is given by $\phi(a_1 + \cdots + a_n p^{n-1}, a_0) = a_0 + a_1 p + \cdots + a_n p^n$.

The Cauchy completion of $\mathbb{N}^*$ with respect to the p-adic metric d_p is $\mathbb{Z}_p$. Thus, we view $\mathbb{N}^*$ as a dense subset of $\mathbb{Z}_p$. In particular, from now on, we will simply write $\mathbb{N}^*$ instead of $(\mathbb{N}^*, d_p)$, with the understanding that it is always viewed as a subspace of the metric space $\mathbb{Z}_p$ with the p-adic metric.

Because ϕ is an isomorphism, $\phi^{-1} : \mathbb{N}^* \longrightarrow F_1\mathbb{N}^*$ is also an isomorphism in Met_1^S and it is given by $\phi^{-1}(a_0 + a_1 p \cdots + a_n p^n) = (a_1 + \cdots + a_n p^{n-1}, a_0)$. We extend ϕ^{-1} to a function from $\mathbb{Z}_p$ to $\frac{1}{p}\mathbb{Z}_p \times V_p$ as follows. Define $\varphi : \mathbb{Z}_p \longrightarrow \frac{1}{p}\mathbb{Z}_p \times V_p (= F_1\mathbb{Z}_p)$ by $\varphi\left(\sum_0^\infty a_i p^i\right) = \left(\sum_1^\infty a_i p^{i-1}, a_0\right)$; i.e., $\varphi((a_i)_{i\geq 0}) = ((a_i)_{i\geq 1}, a_0)$. Then φ is well defined.

Consider $d(\varphi(a_i)_{i\geq 0}, \varphi(b_i)_{i\geq 0}) = \max\{\frac{1}{p} d((a_i)_{i\geq 1}, (b_i)_{i\geq 1}), d(a_0, b_0)\}$. This quantity is 1 or, $\frac{1}{p^j}$ for some $j \geq 1$, provided that $a_0 \neq b_0$ or, $a_i = b_i$, $\forall i \leq j-1$ and $a_j \neq b_j$, respectively. If $a_0 \neq b_0$, then $d(\varphi(a_i)_{i\geq 0}, \varphi(b_i)_{i\geq 0}) = 1 = d((a_i)_{i\geq 0}, (b_i)_{i\geq 0})$. If $a_i = b_i, \forall i \in \{0, \cdots j-1\}$, and $a_j \neq b_j$, where $j \geq 1$, then $d(\varphi(a_i)_{i\geq 0}, \varphi(b_i)_{i\geq 0}) = \frac{1}{p^j} = d((a_i)_{i\geq 0}, (b_i)_{i\geq 0})$.

Thus, φ is an isometry and hence is an isomorphism in the category Met_1^S. Moreover, $(\mathbb{Z}_p, \varphi)$ is a coalgebra of F_1 on Met_1^S.

Theorem 1. The final coalgebra of F_1 on Met_1^S is $(\mathbb{Z}_p, \varphi)$.

Proof. Let (X,e) be any coalgebra of F_1 on Met_1^S. Let $x_0 \in X$. Then $e(x_0)$ is of the form $e(x_0) = (x_1, a_0)$. Similarly $e(x_i) = (x_{i+1}, a_i)$ for $i \geq 1$. Thus, we have sequences $(x_n)_{n\geq 1}$ and $(a_n)_{n\geq 0}$ corresponding to the element $x_0 \in X$. Define $f : X \longrightarrow \mathbb{Z}_p$ by $f(x_0) = (a_i)_{i\geq 0}$. Then $f(x_k) = (a_i)_{i\geq k}$ and it is easy to verify that f makes the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & X \times V_p \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle F_1 f} \\ \mathbb{Z}_p & \xrightarrow{\;\varphi\;} & \mathbb{Z}_p \times V_p \end{array}$$

Since f is uniquely determined by e, f is the unique function that makes this diagram commute (the mediating morphism f is discussed in detail in the next section). To complete the proof, we need to show that f is a short map; i.e., f is a morphism in Met_1^S.

Let $x_0, y_0 \in X$. Then, $e(x_n) = (x_{n+1}, a_n)$, $e(y_n) = (y_{n+1}, b_n)$, where $n = 0,1,2,\cdots$. Moreover, $f(x_0) = (a_i)_{i\geq 0}$ and $f(y_0) = (b_i)_{i\geq 0}$. Note that $d(f(x_0), f(y_0)) = d((a_i)_{i\geq 0}, (b_i)_{i\geq 0})$ takes the value 0 or 1 or $\frac{1}{p^j}$ provided that $a_i = b_i, \forall i$ or $a_0 \neq b_0$ or $a_i = b_i, \forall i \leq j-1$ and $a_j \neq b_j$ respectively. If $a_i = b_i$ for all $i \geq 0$, then there is nothing to prove.

Consider the case $a_0 \neq b_0$. Then, $d(f(x_0), f(y_0)) = d((a_i)_{i\geq 0}, (b_i)_{i\geq 0}) = 1$. Since e is a short map, $d(e(x_0), e(y_0)) \leq d(x_0, y_0)$. However $d(e(x_0), e(y_0))$ is equal to $\max\{\frac{1}{p} d(x_1, y_1), d(a_0, b_0)\}$ which takes the value 1 as $d(a_0, b_0) = 1$. Therefore, $1 \leq d(x_0, y_0)$. Since d is a one-bounded metric, $d(x_0, y_0) = 1$. Thus, $d(f(x_0), f(y_0))\, d((a_0)_{i\geq 0}, (b_0)_{i\geq 0}) = 1 = d(x_0, y_0)$.

Now consider the case $a_i = b_i, \forall i \leq j-1$ and $a_j \neq b_j$. Then, $d(f(x_0), f(y_0)) = d((a_0)_{i\geq 0}, (b_0)_{i\geq 0}) = \frac{1}{p^j}$.

Consider the following chain iterated from the coalgebra morphism $X \xrightarrow{e} F_1 X$.

$$X \xrightarrow{e} F_1 X \xrightarrow{F_1 e} F_1^2 X \cdots \xrightarrow{F_1^{j-1} e} F_1^j X \xrightarrow{F_1^j e} F_1^{j+1} X$$

We have $F_1^j e \circ F_1^{j-1} e \circ \cdots e(x_0) = (x_{j+1}, a_j, a_{j-1}, \cdots a_0)$ and $F_1^j e \circ F_1^{j-1} e \circ \cdots e(y_0) = (y_{j+1}, b_j, b_{j-1}, \cdots b_0)$. Because $d(a_{j-1}, b_{j-1}) = d(a_{j-2}, b_{j-2}) = \cdots = d(a_0, b_0) = 0$ as $a_i = b_i$ for all $i \leq j-1$ and $a_j \neq b_j$,

$$d\left(F_1^j e \circ F_1^{j-1} e \circ \cdots e(x_0), F_1^j e \circ F_1^{j-1} e \circ \cdots e(y_0)\right)$$
$$= \max\{\frac{1}{p^{j+1}} d(x_{j+1}, y_{j+1}), \frac{1}{p^j} d(a_j, b_j),$$
$$\frac{1}{p^{j-1}} d(a_{j-1}, b_{j-1}), \cdots, d(a_0, b_0)\}$$
$$= \frac{1}{p^j}.$$

Moreover, $d\left(F_1^j e \circ F_1^{j-1} e \circ \cdots e(x_0), F_1^j e \circ F_1^{j-1} e \circ \cdots e(y_0)\right) \leq d(x_0, y_0)$, because $F_1^j e \circ F_1^{j-1} e \circ \cdots \circ e$ is a short map. Thus, we have $\frac{1}{p^j} \leq d(x_0, y_0)$ and hence $d(f(x_0), f(y_0)) = \frac{1}{p^j} \leq d(x_0, y_0)$.

Therefore, f is a short map as required. This completes the proof. □

The final coalgebra of F_1 on Met_1^S is the Cauchy completion of the initial algebra

First, let us briefly recall the Cauchy completion of a metric space. Let (X, d) be a metric space. Then there exists a complete metric space $(\mathbf{C}(X), d)$ and an isometric embedding $\nu_X : X \to \mathbf{C}(X)$ such that $\overline{\nu_X(X)} = \mathbf{C}(X)$. For a metric space X, $\mathbf{C}(X)$ can be constructed as follows. Consider the equivalence relation $\sim$ on the set S of all Cauchy sequences in X given by $(a_n) \sim (b_n)$ if $\lim_{n\to\infty} d(a_n, b_n) = 0$.

The quotient set $S/\sim$ of equivalence classes is the required complete metric space $\mathbf{C}(X)$, where the metric d on $\mathbf{C}(X)$ is given by $d([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n)$. The required isometric embedding ν_X of X into $\mathbf{C}(X)$ is the one which sends every element x in X to the equivalence class $[(x)]$ of the constant sequence $(x_n = x)_n$.

The above construction can be extended to an endofunctor C on the category of metric spaces. Specifically, $\mathbf{C}(f : X \to Y) = \mathbf{C}(f) : \mathbf{C}(X) \to \mathbf{C}(Y)$, where $\mathbf{C}(f)$ is given by $\mathbf{C}(f)([(x_n)]) = [(f(x_n))]$. It directly follows from the construction that if f is a short map, then so is $\mathbf{C}(f)$. Thus, the Cauchy completion gives rise to an endofunctor C: $Met_1^S \to Met_1^S$.

Fix a metric space X. Because V_p is finite, for each sequence (x_n, b_n) in $\frac{1}{p} X \times V_p$, some element in V_p must occur infinitely many times in the sequence (b_n). Let b be such an element in V_p and (x_{n_k}) be the subsequence of (x_n), such that the corresponding second coordinates of the subsequence (x_{n_k}, b_{n_k}) are all b. Define $\alpha_X : \mathbf{C}(F_1(X)) = \mathbf{C}\left(\frac{1}{p}X \times V_p\right) \to F_1(\mathbf{C}(X)) = \frac{1}{p}\mathbf{C}(X) \times V_p$ by $\alpha_X([(x_n, b_n)]) = ([(x_{n_k})], b)$. One can then check that α_X is well defined [see also (Bhattacharya *et al*., 2014)] and that the collection of morphisms $\alpha = \{\alpha_X\}$ determines a natural isomorphism between $C \circ F_1$ and $F_1 \circ C$.

The inverse of α_X is the morphism $\beta_X : \frac{1}{p}\mathbf{C}(X) \times V_p \to \mathbf{C}(\frac{1}{p}X \times V_p)$ given by $\beta_X([(x_n)], a) = [(x_n, a)]$. Hence, the collection of maps $\beta = \{\beta_X\}$ is the inverse of α. Therefore, all α_X's and β_X's are isometric embeddings and in particular short maps as required.

Consider the initial algebra of F_1. As the associated morphism is an isomorphism, by reversing the arrow we have the coalgebra $(\mathbb{N}^*, \phi^{-1} : \mathbb{N}^* \to \frac{1}{p}\mathbb{N}^* \times V_p)$. By applying the Cauchy completion functor to this coalgebra, we have $\mathbf{C}(\phi^{-1}) : \mathbf{C}(\mathbb{N}^*) \to \mathbf{C}(\frac{1}{p}\mathbb{N}^* \times V_p)$. After composing with $\alpha_{\mathbb{N}^*}$, we have a coalgebra $\alpha_{\mathbb{N}^*} \circ \mathbf{C}(\phi^{-1}) : \mathbf{C}(\mathbb{N})^* \to \frac{1}{p}\mathbf{C}(\mathbb{N}^*) \times V_p$.

For any $(a_i) \in \mathbb{Z}_p$, the sequence (x_n) in $\mathbb{N}^*$ given by $x_n = \sum_0^n a_i p^i$, is Cauchy. It follows that $f : \mathbb{Z}_p \to C(\mathbb{N}^*)$ defined by $f((a_i)_i) = [(\sum_{j=0}^n a_j p^j)_n]$ is a well defined isometric embedding. Note that if $x = \sum_0^k a_i p^i = (a_0, ..., a_k, 0, \cdots) \in \mathbb{N}^*$, then $f(x)$ is the equivalence class of the constant sequence (x). Therefore, $f(\mathbb{N}^*) = \nu_{\mathbb{N}^*}(\mathbb{N}^*)$. Hence, $f(\mathbb{N}^*)$ is dense in $\mathbf{C}(\mathbb{N}^*)$. Since f is isometric, it also follows that $f(\mathbb{Z}_p) = \mathbf{C}(\mathbb{N}^*)$. Thus, f is an isomorphism in the category Met_1^S.

Now, consider the following diagram.

$$\begin{array}{ccc} \mathbf{C}(\mathbb{N}^*) & \xleftarrow{\quad f \quad} & \mathbb{Z}_p \\ {\scriptstyle \alpha_{\mathbb{N}^*} \circ \mathbf{C}(\phi^{-1})} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \frac{1}{p}C(\mathbb{N}^*) \times V_p & \xleftarrow[F_1 f]{} & \frac{1}{p}\mathbb{Z}_p \times V_p \end{array}$$

Because,

$$\begin{aligned} \left(\alpha_{\mathbb{N}^*} \circ \mathbf{C}(\phi^{-1}) \circ f\right)\left(\sum_0^\infty a_i p^i\right) &= \left(\alpha_{\mathbb{N}^*} \circ \mathbf{C}(\phi^{-1})\right)\left([(\sum_0^n a_i p^i)_n]\right) \\ &= \alpha_{\mathbb{N}^*}([(\sum_1^n a_i p^{i-1}, a_0)_n]) \\ &= ([(\sum_1^n a_i p^{i-1})_n], a_0) \\ &= F_1 f(\sum_1^\infty a_i p^{i-1}, a_0) \\ &= (F_1 f \circ \varphi)\left(\sum_0^\infty a_i p^i\right) \end{aligned}$$

f makes the above diagram commute. It follows that f is an isomorphism between the F_1-coalgebras $(C(\mathbb{N}^*), \alpha_{\mathbb{N}^*} \circ C\phi^{-1})$ and $(\mathbb{Z}_p, \varphi)$. As the right column is the final coalgebra (Theorem 1), we have proved the following theorem.

Theorem 2. $(\mathbf{C}(\mathbb{N}^*), \alpha_{\mathbb{N}^*} \circ \mathbf{C}(\phi^{-1}))$ is the final coalgebra of F_1 *on* Met_1^S, which is the Cauchy completion of the initial algebra.

Thus, Theorem 2 provides a method to obtain the final coalgebra from the initial algebra by Cauchy completion.

FINAL COALGEBRA OF F_1 ON Met_1^C AND Set_1

Consider F_1 on Set_1. As described in the previous section, one can easily prove the following result in a similar way to the proof of Lemma 2.

Lemma 3. The initial algebra of F_1 on Set_1 is the pair $(\mathbb{N}^*, \phi)$, where $\phi : F_1\mathbb{N}^* \longrightarrow \mathbb{N}^*$ is given by $\phi(a_1 + \cdots + a_n p^{n-1}, a_0) = a_0 + a_1 p \cdots + a_n p^n$.

It can also be shown using Barr's sufficient condition [see Proposition 1.3 of (Barr, 1993)], that the final coalgebra of F_1 on Set_1 is $(\mathbb{Z}_p, \varphi)$, leaving out the metric structure of $\mathbb{Z}_p$.

Lemma 4. The final coalgebra of F_1 on Set_1 is the pair $(\mathbb{Z}_p, \varphi)$, where $\varphi : \mathbb{Z}_p \longrightarrow \frac{1}{p}\mathbb{Z}_p \times V_p (= F_1\mathbb{Z}_p)$ is given by $\varphi((a_i)_{i \geq 0}) = ((a_i)_{i \geq 1}, a_0)$.

Remark 1. Note that the final coalgebra and the initial algebra of F_1 on Set_1 can be obtained by forgetting the metric structure of the final coalgebra and the initial algebra of F_1 on Met_1^S, respectively.

In the rest of this section, we will consider F_1 on Met_1^C and its final coalgebra. Motivated by the observation in the above remark, one may question whether the final coalgebra of F_1 on Met_1^C is the same as that of F_1 on Met_1^S. We will show here that this is indeed the case.

Consider F_1 on Met_1^C. In a previous section, we identified the initial algebra of the endofunctor F_1 on Met_1^S, and the final coalgebra of F_1 on Met_1^S was obtained by Cauchy completing the initial algebra. Recall that the initial algebra and the final coalgebra of F_1 on Set_1 are the same as those of F_1 on Met_1^S, but without the metric structure. One can make use of this fact to compute the mediating morphism from a given co-algebra to the final coalgebra. We will show that if the coalgebra morphism is continuous, then the mediating morphism is also continuous.

Let (X,e) be a coalgebra of F_1 on Met_1^C. Then, after forgetting the metric structure, (X,e) is an F_1–coalgebra at the set level. Since $(\mathbb{Z}_p, \varphi)$ is the final coalgebra of F_1 on Set_1, there exists a unique map $g : X \longrightarrow \mathbb{Z}_p$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{e} & X \times V_p \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle F_1 g} \\ \mathbb{Z}_p & \xrightarrow{\varphi} & \mathbb{Z}_p \times V_p \end{array}$$

By applying F_1 to the coalgebra $X \xrightarrow{e} X \times V_p$ repeatedly, we have the following chain.

$$X \xrightarrow{e} X \times V_p \xrightarrow{F_1 e} X \times V_p \times V_p \cdots$$
$$\longrightarrow X \times V_p \cdots \times V_p \xrightarrow{F_1^{n-1} e} X \times V_p \cdots \times V_p$$

Let $x_0 \in X$. Then, $e(x_0)$ is of the form $e(x_0) = (x_1, a_0)$ and similarly $e(x_1) = (x_2, a_1)$. Inductively, for each $n \in \mathbb{N}$, let $e(x_n) = (x_{n+1}, a_n)$, where $a_i \in V_p$ for all $i \in \{0, 1, \cdots\}$. Consider the sequences (χ_n) of elements in $\mathbb{N}^*$, where $\chi_n = a_0 + a_1 p + \cdots + a_n p^n$. We claim that (χ_n) is a Cauchy sequence in $\mathbb{Z}_p$.

Let $m > n$ be integers. Then, $d(\chi_n, \chi_m)$ takes the value $\frac{1}{p^k}$, where $n+1 \leq k \leq m$, or zero. If it takes the value zero then there is nothing to prove. Consider the non-zero case. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ large enough so that $\frac{1}{p^N} < \epsilon$. For $m > n > N$, $d(\chi_n, \chi_m) = \frac{1}{p^k} \leq \frac{1}{p^{n+1}} < \frac{1}{p^N} < \epsilon$. Thus, (χ_n) is a Cauchy sequence as claimed.

As $\mathbb{Z}_p$ is complete and (χ_n) is Cauchy, $\lim_{n\to\infty} \chi_n$ exists in $\mathbb{Z}_p$. Define $f : X \longrightarrow \mathbb{Z}_p$ by $f(x_0) = \lim_{n\to\infty} \chi_n$. Then, one can easily show that f is well defined and the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{e} & X \times V_p \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F_1 f} \\ \mathbb{Z}_p & \xrightarrow{\varphi} & \mathbb{Z}_p \times V_p \end{array}$$

Thus f defined above is the mediating morphism at the set level.

Proposition 1. The mediating morphism f, defined above, is continuous.

Proof. We will show the continuity of f at some fixed $x \in X$. Let $\epsilon > 0$ and $y \in X$. From the definition of f we have $d_{\mathbb{Z}_p}(f(x), f(y)) = \lim_{n\to\infty} d_{\mathbb{N}^*}(\chi_n, \chi_n')$. Therefore,

there is some $N \in \mathbb{N}$ such that $d_{\mathbb{Z}_p}(f(x), f(y)) - \frac{\epsilon}{2} < d_{\mathbb{N}^*}(\chi_N, \chi_N')$

Now the function $g_N = F_1^N e \circ F_1^{N-1} \circ \cdots \circ F_1 e \circ e$ is continuous on X as e and all $F_1^i e$'s are continuous. Since g_N is continuous at x, $\exists \delta_N > 0$ such that $d(g_N(x), g_N(y)) < \frac{\epsilon}{2}$ whenever $d(x, y) < \delta_N$.

Suppose that $d(x, y) < \delta_N$. Then we have

$$d(g_N(x), g_N(y)) = \max\{\frac{d(x_{N+1}, y_{N+1})}{p^{N+1}}, \quad \frac{d(a_N, b_N)}{p^N}, \cdots \frac{d(a_1, b_1)}{p}, d(a_0, b_0)\}$$
$$< \frac{\epsilon}{2}$$

If $a_0 \neq b_0$, then $\max\{\frac{d(x_{N+1}, y_{N+1})}{p^{N+1}}, \frac{d(a_N, b_N)}{p^N}, \cdots \frac{d(a_1, b_1)}{p}, d(a_0, b_0)\} = 1 < \frac{\epsilon}{2}$, which is not possible. Thus, there must be some $j \geq 1$ such that $a_i = b_i$, $\forall i \leq j - 1$ and $a_j \neq b_j$. Consider the cases $j \leq N$ and $j > N$.

When $j \leq N$:

$$d(g_N(x), g_N(y)) = \max\left\{\frac{d(x_{N+1}, y_{N+1})}{p^{N+1}}, \frac{d(a_N, b_N)}{p^N}, \cdots \frac{d(a_1, b_1)}{p}, d(a_0, b_0)\right\}$$
$$= \frac{1}{p^j} < \frac{\epsilon}{2}$$

Thus, we have $d_{\mathbb{Z}_p}(f(x), f(y)) - \frac{\epsilon}{2} < d_{\mathbb{N}^*}\left(\chi_N, \chi_N'\right) = \frac{1}{p^j} < \frac{\epsilon}{2}$. It follows that $d_{\mathbb{Z}_p}(f(x), f(y)) < \epsilon$.

When $j > N$: In this case $d_{\mathbb{Z}_p}(f(x), f(y)) - \frac{\epsilon}{2} < d_{\mathbb{N}^*}\left(\chi_N, \chi_N'\right) = 0$. Thus, $d_{\mathbb{Z}_p}(f(x), f(y)) < \epsilon$.

This completes the proof $F_1 f$. □

The mediating morphism is uniquely determined for a given coalgebra (X,e) with continuous e. Therefore, $(\mathbb{Z}_p, \varphi)$ is the final coalgebra of F_1 on Met_1^C.

Corollary 1. The final coalgebra of F_1 on Met_1^C is the same as that of F_1 on Met_1^S.

We may ask whether a similar result as Corollary 1 holds for the initial algebra of F_1. More specifically; Is $(\mathbb{N}^*, \phi)$ the initial algebra of F_1 on Met_1^C? Though a similar result holds for the initial algebra of F_1 on Met_1^S and Set_1 (see Remark 1), we answer this question negatively in the following section by giving a counter example.

On the initial algebra of F_1 on Met_1^L and Met_1^C

Recall that the initial algebra of F_1 on Met_1^S is $(\mathbb{N}^*, \phi)$, where $\phi : F_1\mathbb{N}^* \longrightarrow \mathbb{N}^*$ is given by $\phi(a_1 + a_2p + \cdots + a_kp^{k-1}, a_0) = a_0 + a_1p + \cdots + a_kp^k$. The pair $(\mathbb{N}^*, \phi)$ is also the initial algebra of F_1 on Set_1 leaving out the metric structure of $\mathbb{N}^*$.

Let (X,e) be an F_1 algebra at the set level. Then, there exists a unique function $f : \mathbb{N}^* \to X$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{N}^* \times V_p & \xrightarrow{\phi} & \mathbb{N}^* \\ {\scriptstyle F_1 f}\downarrow & & \downarrow{\scriptstyle f} \\ X \times V_p & \xrightarrow{e} & X \end{array}$$

Since ϕ is an isomorphism, $\phi^{-1} : \mathbb{N}^* \to \mathbb{N}^* \times V_p$ exists and is given by,

$$\phi^{-1}(a_0 + a_1p + \cdots + a_kp^k) = (a_1 + a_2p + \cdots + a_kp^{k-1}, a_0).$$

Now the diagram at the bottom of the page is obtained by applying F_1 to the above diagram repeatedly, with the top horizontal arrow reversed.

Any number in $\mathbb{N}^*$ can be written as a polynomial in p with coefficients in V_p. Let $a_0+a_1p+\cdots+a_kp^k \in \mathbf{N}^*$ be arbitrary. Then, $F_1^k\phi^{-1}\circ F_1^{k-1}\phi^{-1}\circ\cdots\circ\phi^{-1}(a_0+a_1p+\cdots+a_kp^k) = (0, a_k, a_{k-1}, \cdots, a_0)$ and $F_1^{k+1}f(0, a_k, a_{k-1}, \cdots, a_0) = (*, a_k, a_{k-1}, \cdots, a_0)$. Since the above diagram commutes, we can write f as follows.

$$f(a_0 + a_1p + \cdots + a_kp^k) = e \circ F_1e \circ \cdots F_1^k e(*, a_k, a_{k-1}, \cdots, a_0) \quad (1)$$

Now, consider the following example. Let $X_0 = \{0, 1\}$ be a one-ointed metric space with the distinguished element 0 and the discrete metric. Define $e : X_0 \times V_p \longrightarrow X_0$ by $e(0, 0) = 0,\ e(0, a) = 1$ if $a \neq 0$ and $e(1, a) = 1$, $\forall a \in V_p$. Then, e is a Lipschitz map with the Lipschitz constant p. In particular e is continuous. Thus, (X_0,e) is an algebra of F_1 on Met^L_1 and on Met_1^C . Further, (X_0,e) is an algebra of F_1 on Set_1, after leaving out the metric structure.

We claim that $(\mathbb{N}^*, \phi)$ is not the initial algebra of F_1 on Met_1^C . Towards a contradiction, suppose that $(\mathbb{N}^*, \phi)$ is the initial algebra of F_1 on Met_1^C . Then, there exists a unique continuous map $f : \mathbb{N}^* \longrightarrow X_0$ such that the required square commutes.

If we leave out the metric structure , then f is the unique function such that the diagram at the set level commutes, as $(\mathbb{N}^*, \phi)$ is the initial algebra of F_1 on Set_1. Recall that f is given by equation (1). Since f is continuous at 0, there must be some $\delta > 0$ such that $d(f(0), f(y)) < \frac{1}{2}$ for every y such that $d(0,y) < \delta$. Choose n large enough so that $\dfrac{1}{p^n} < \delta$. Then, $d(0, p^{n+1}) = \dfrac{1}{p^{n+1}} < \delta$. Consider the case when $y = p^{n+1}$.

Then,

$$\begin{aligned} f(p^{n+1}) &= f(0 + 0.p + 0.p^2 + \cdots + 1.p^{n+1}) \\ &= e \circ F_1e \circ \cdots F_1^{n+1}e(0, 1, 0, \cdots, 0) \\ &= e \circ F_1e \circ \cdots F_1^{n}e(1, 0, \cdots, 0) \\ &= e \circ F_1e \circ \cdots F_1^{n-1}e(1, 0, \cdots, 0) \\ &= \ \vdots \quad \vdots \quad \vdots \\ &= e(1, 0) = 1. \end{aligned}$$

Because $f(0) = 0\,, f(y) = 1,$ we have $d(f(0), f(y)) = 1 < \frac{1}{2}$, which is a contradiction.

$$\begin{array}{ccccccccc} F_1^{k+1}\mathbb{N}^* & \xleftarrow[F_1^k\phi^{-1}]{} & F_1^k\mathbb{N}^* & \xleftarrow[F_1^{k-1}\phi^{-1}]{} & \cdots \mathbb{N}^* \times V_p \times V_p & \xleftarrow[F_1\phi^{-1}]{} & \mathbb{N}^* \times V_p & \xleftarrow[\phi^{-1}]{} & \mathbb{N}^* \\ {\scriptstyle F_1^{k+1}f}\downarrow & & {\scriptstyle F_1^k f}\downarrow & & {\scriptstyle F_1^2 f}\downarrow & & {\scriptstyle F_1 f}\downarrow & & \downarrow{\scriptstyle f} \\ F_1^{k+1}X & \xrightarrow{F_1^k e} & F_1^k X & \xrightarrow{F_1^{k-1}e} & \cdots X \times V_p \times V_p & \xrightarrow{F_1 e} & X \times V_p & \xrightarrow{e} & X \end{array}$$

Therefore $(\mathbb{N}^*, \phi)$ is not the initial algebra of F_1 on Met_1^C. The same example also implies that $(\mathbb{N}^*, \phi)$ is not the initial algebra of F_1 on Met_1^L.

Proposition 2. The initial algebra of F_1 on Met_1^S is neither the initial algebra of F_1 on Met_1^L nor the initial algebra of F_1 on Met_1^C.

REFERENCES

Adámek J. (1974). Free algebras and automata realizations in the language of categories. *Commentationes Mathematicae Universitatis Carolinae* **15**(4): 589–602.

Adámek J., Milius S. & Moss L. (2010). Initial algebras and terminal coalgebras: a survey. *Draft, June*, 29:18.

Barr M. (1993). Terminal coalgebras in well-founded set theory. *Theoretical Computer Science* **114**(2):299–315.

Bhattacharya P. (2015). The p-adic integers as final coalgebra. Logic, language, information and computation (eds. V. de Paiva, R. de Queiroz, L. Moss, D. Leivant & A. de Oliveira). WoLLIC 2015. *Lecture Notes in Computer Science*, volume 9160, pp. 189–199. Springer, Berlin, Heidelberg, Germany. DOI: https://doi.org/10.1007/978-3-662-47709-0_14

Bhattacharya P., Moss L., Ratnayake J. & Rose R. (2014). Fractal sets as final coalgebras obtained by completing an initial algebra. In: *Horizon of the Mind, Essays Dedicated to Prakash Panangen on the Occasion of His 60 Birthday* (eds. F. Van Breugel, E. Kashafi, C. Palamidessi & J. Rutten), pp. 146–167. Springer, Berlin, Germany.

Cuoco A.A. (1991). Visualizing the p-adic integers. *The American Mathematical Monthly* **98**(4): 355–364. DOI: https://doi.org/10.1080/00029890.1991.12000770

Hutchinson J.E. (1981). Fractals and self similarity. *Indiana University Mathematics Journal* **30**(5): 713–747. DOI: https://doi.org/10.1512/iumj.1981.30.30055

Leinster T. (2011). A general theory of self-similarity. *Advances in Mathematics* **226**(4): 2935–3017. DOI: https://doi.org/10.1016/j.aim.2010.10.009

Moss L.S., Ratnayake J. & Rose R. (2013). Fractal sets as final coalgebras obtained by completing an initial algebra. *Topology, Algebra, and Categories in Logic*, page 157.

Serre J.-P. (2012). *A Course in Arithmetic*, volume 7. Springer Science & Business Media, Berlin, Germany.