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RESEARCH ARTICLE

Bayesian inference from the mixture of half-normal distributions under censoring

Tabassum Naz Sindhu[1,2*], Hafiz M.R. Khan[3], Zawar Hussain[1] and Bander Al-Zahrani[4]
[1] *Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan.*
[2] *Department of Sciences and Humanities, FAST - National University, Islamabad, Pakistan.*
[3] *Department of Public Health, Texas Tech University Health Sciences Center, Lubbock, TX 79430, USA.*
[4] *Department of Statistics, King Abdulaziz University, Saudi Arabia.*

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Abstract: This study considers the Bayesian inference for the mixture of two components of half-normal distribution using non-informative and informative prior. Several of its structural properties were derived, including explicit expression for mean, median, mode, reliability and hazard rate functions. Due to cost and time constraints, in most lifetime testing experiments censoring is an obligatory feature of lifetime datasets. We investigated Bayesian estimation of the parameters using various loss functions. The prior belief of the mixture model is represented by the uniform and square-root inverted gamma priors. Some properties of the model with graphs of the mixture density and hazard function are also discussed. The efficiencies of the proposed set of estimates of the mixture model parameters were studied through simulation and a real life dataset. Posterior risks of the Bayes estimators are evaluated and compared to explore the effect of prior belief and loss functions.

Keywords: Half-normal distribution, hazard rate function, loss functions, mixture distribution, posterior risk, reliability function.

INTRODUCTION

There is a great need for efficient estimation of mixture distribution, especially following the explosion in the use of modelling tools in many applied fields, such as engineering, medicine, and other sciences. In almost every field mixture models are used to model diverse populations. Mixture distributions are the best choice for a statistical population containing two or more subpopulations. Mixture densities can be used to model a statistical population with subpopulations. Mixture components form the densities of the subpopulations and the weights serve as the proportion of each subpopulation in the complete population. There have been a number of studies related to statistical inference based on mixture distribution. Sindhu *et al*. (2014a) studied Bayesian analysis of the shape parameter of the mixture of Burr type X distribution using censored data. Gosh and Ebrahimi (2001) have studied the Bayesian analysis of the mixing function in a mixture of two exponential distributions. Saleem and Aslam (2008) presented a comparison study of the maximum likelihood estimates with the Bayes estimates assuming the uniform and the Jeffreys priors for the parameters of the Rayleigh mixture. Sindhu *et al*. (2014b) have considered the Bayesian inference for mixture Burr type II distribution under type-I censoring. Saleem *et al*. (2010) considered the Bayesian analysis of the mixture of power function distribution using the complete and censored sample. Sultan *et al*. (2007) discussed the properties of the two component mixture of inverse Weibull distribution under classical approach. Sindhu *et al*. (2014c) studied the Bayes estimation of the parameters of a mixture of two Rayleigh distributions under doubly censoring. With highly reliable components, it is unusual if all the components have failed by the end of the time allocated for the test. When all the subjects are scheduled to begin the study at the same time and end the study at the same time, type I censoring occurs. Type I censoring is usually used in survival studies and in some engineering studies.

* Corresponding author (sindhuqau@gmail.com; https://orcid.org/0000-0001-9433-4981)

The half-normal distribution has been used for modelling positively skewed data. The most popular model used to describe the lifetime process under fatigue is the half-normal distribution. The half-normal model has been studied by several authors. Bland and Altman (1999) have studied the half-normal model for dealing with the relationships between measurement and magnitude error. Cohen (1991) studied the problem of the inference of truncated distributions, including the truncated normal through the classical approach. The classical inference for half-normal model was examined by Pewsey (2002; 2004). Cooray and Ananda (2008) defined the generalised half-normal distribution derived from a model for static fatigue. Gauss *et al*. (2012) studied the Kumaraswamy generalised half-normal distribution for modelling skewed positive data. Wiper *et al*. (2008) have studied the Bayesian inference for half-normal and half-t distribution under non-informative priors. Bayes estimation of Gumbel mixture models with industrial applications was considered by Sindhu *et al.* (2016a). A study of a cumulative quantity control chart for a mixture of Rayleigh model under Bayesian framework was studied by Sindhu *et al.* (2016b).

The present study investigated the prominent features of the mixture of half-normal distribution. The mixture of this type has not been studied earlier in literature through Bayesian structure, to the best of our knowledge. In a study by Leiva *et al*. (2009) guinea pigs were injected with different doses of tubercle bacilli. It can be considered that guinea pigs with a low dose may follow a half-normal distribution for a specific parameter. If the dose is modified to an above optimal level, the survival time may also follow the half-normal distribution, but at a lower survival time. Due to the difference in the level of doses, the weight variability in the distribution is created. In this case, a mixture of half-normal distribution occur. Similarly, a mixture of half-normal distribution can be used to investigate the relationship between stress and student exam scores. The lower stress may contribute to half-normal distribution, and higher stress may produce a similar distribution with different weight variation in the parameter. Their combination can generate a mixture of half-normal distribution.

The aim of the current study was to present the general form of the mixture model, with the properties and likelihood function of the mixture of half-normal distributions. Inferential procedures with Bayes estimator were considered for the set of parameters, including the posterior distribution, Bayes estimates and posterior risks under different loss functions. The simulation study with detailed comparison of the estimates and numerical examples are given.

METHODOLOGY

Mixture model and its properties

A finite mixture distribution with m-component densities of specified parametric form and unknown mixing weight p_i is given by:

$$f_z(z)=\sum_{i=1}^{m} p_i f_i(z),\ 0<p_i\le 1,\ \sum_{i=1}^{m} p_i=1.$$

The half-normal distribution is supposed for m-components of the mixture:

$$f_i(x;\lambda_i)=\sqrt{\frac{2}{\pi}}\lambda_i^{-1}\exp\left(-\frac{x^2}{2\lambda_i^2}\right);\lambda_i>0,\ x>0,\ \text{and } i=1,2,\ldots,m, \quad ...(1)$$

where λ is the scale parameter. Thus, the mixture model is of the following form:

$$f(x;\Delta)=\sum_{i=1}^{m} p_i\sqrt{\frac{2}{\pi}}\lambda_i^{-1}\exp\left(-\frac{x^2}{2\lambda_i^2}\right);\ i=1,2,\ldots,m,\ \sum_{i=1}^{m} p_i=1.$$

The distribution function of the corresponding mixture distribution is as follows:

$$\begin{aligned}F(x;\Delta)&=p_1F_1(x;\Delta)+p_2F_2(x;\Delta)+\ldots+p_mF_m(x;\Delta)\\&=\sum_{i=1}^{m} p_i erf\left(\frac{x}{\sqrt{2}\lambda_i}\right),\ \ i=1,2,\ldots,m,\ \sum_{i=1}^{m} p_i=1,\end{aligned}$$

where

$\Delta=\{(p_1,p_2,\ldots,p_m),(\lambda_1,\lambda_2,\ldots,\lambda_m)\}$, and $\Delta_i=(\lambda_i,p_i)$, $i=1,2,\ldots,m$, and $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-u^2)\,du$, is the error function. The error function can be conveniently expressed in terms of other functions and series as follows:

$$\operatorname{erf}(x)=\frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2},x^2\right)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)}}{n!(2n+1)},$$

where $\gamma(\cdot)$ is the incomplete gamma function. The main focus is to study the mixture model for m = 2. Graphical representations for the selected parametric values for the two components mixture model are shown in Figure 1.

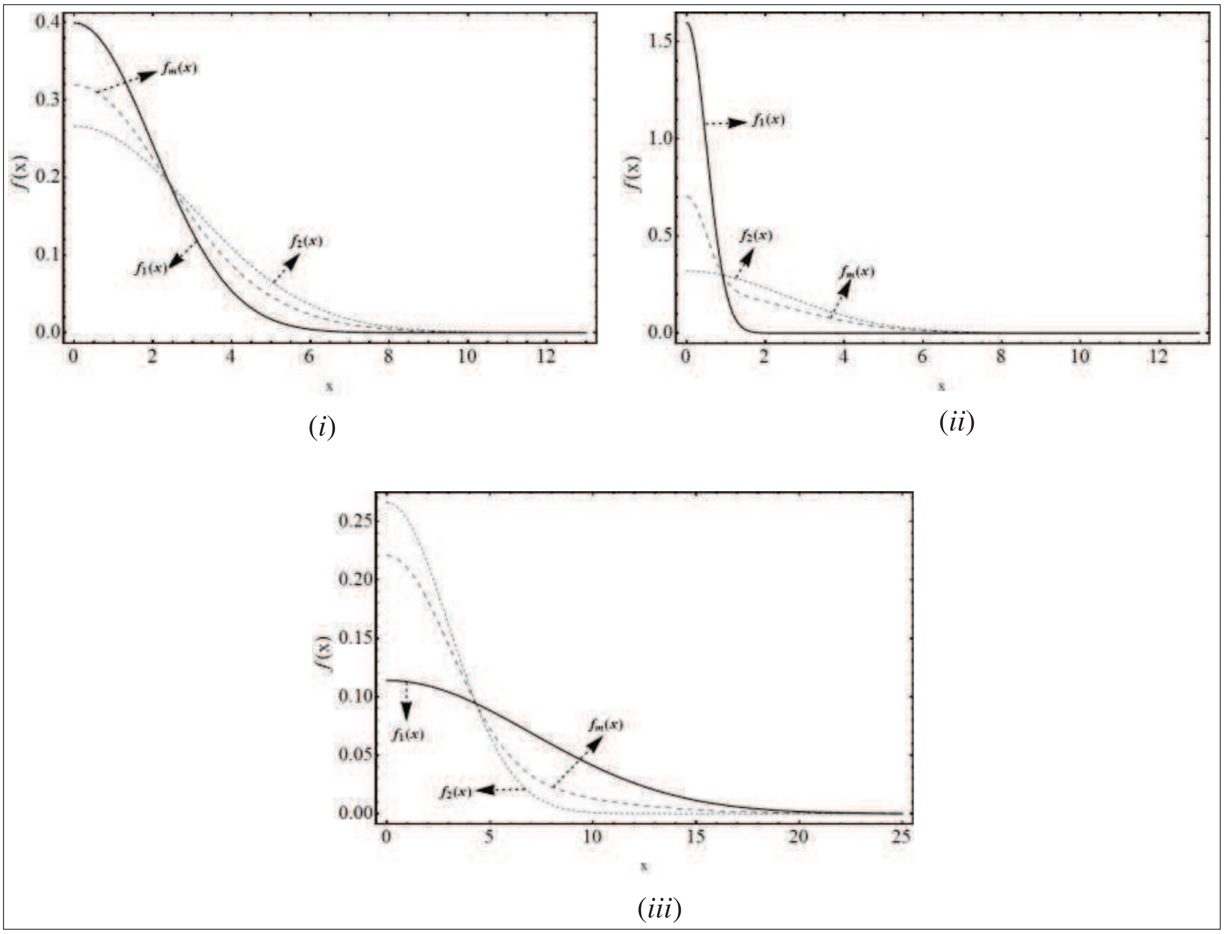

Figure 1: Density function of components and their mixtures $(p_1, \lambda_1, \lambda_2)$: (*i*) (0.4, 2, 3); (*ii*) (0.3, 0.5, 2.5); (*iii*) (0.3, 0.5, 7)

Reliability function

The reliability function or survival function of the two components mixture of half-normal distribution is given by:

$$R(t) = p_1\left\{1 - \text{erf}\left(\frac{t}{\sqrt{2}\lambda_1}\right)\right\} + p_2\left\{1 - \text{erf}\left(\frac{t}{\sqrt{2}\lambda_2}\right)\right\}.$$

Failure rate function

The failure rate function (hazard rate function) of the two components mixture of half-normal distribution is given by:

$$r(t) = \frac{p_1\sqrt{\frac{2}{\pi}}\lambda_1^{-1}\exp\left(-\frac{t^2}{2\lambda_1^2}\right) + p_2\sqrt{\frac{2}{\pi}}\lambda_2^{-1}\exp\left(-\frac{t^2}{2\lambda_2^2}\right)}{p_1\left\{1 - \text{erf}\left(\frac{t}{\sqrt{2}\lambda_1}\right)\right\} + p_2\left\{1 - \text{erf}\left(\frac{t}{\sqrt{2}\lambda_2}\right)\right\}},$$

which can be written considering the result of Sultan *et al*. (2007) as $r(t) = h(t)r_1(t) + \{1 - h(t)\}r_2(t)$. The derivative of hazard rate function is given as:

$$r'(t) = h(t)r_1'(t) + \{1 - h(t)\}r_2'(t) - h(t)\{1 - h(t)\}\{r_1(t) - r_2(t)\}^2,$$

where $h(t) = \left(1 + \frac{p_2 R_2(t)}{p_1 R_1(t)}\right)^{-1}$, $r_i(t) = \frac{f_i(t)}{R_i(t)}$ and $i = 1, 2$.

The failure rate function of the two components mixture of half-normal distribution satisfies the following limits;

$\lim_{t\to\infty} h(t) = \frac{p_1}{p_1 + p_2} = p_1$, and $\lim_{t\to\infty} \frac{p_2 R_2(t)}{p_1 R_1(t)} = \frac{p_2}{p_1} \neq -1$. It follows that $h(t) < \infty$. It can be shown that

$$\lim_{t\to 0} r_i(t) = \sqrt{\frac{2}{\pi}}\lambda^{-1} \text{ and } \lim_{t\to\infty} r_i(t) = 0 \text{ for } i = 1, 2.$$

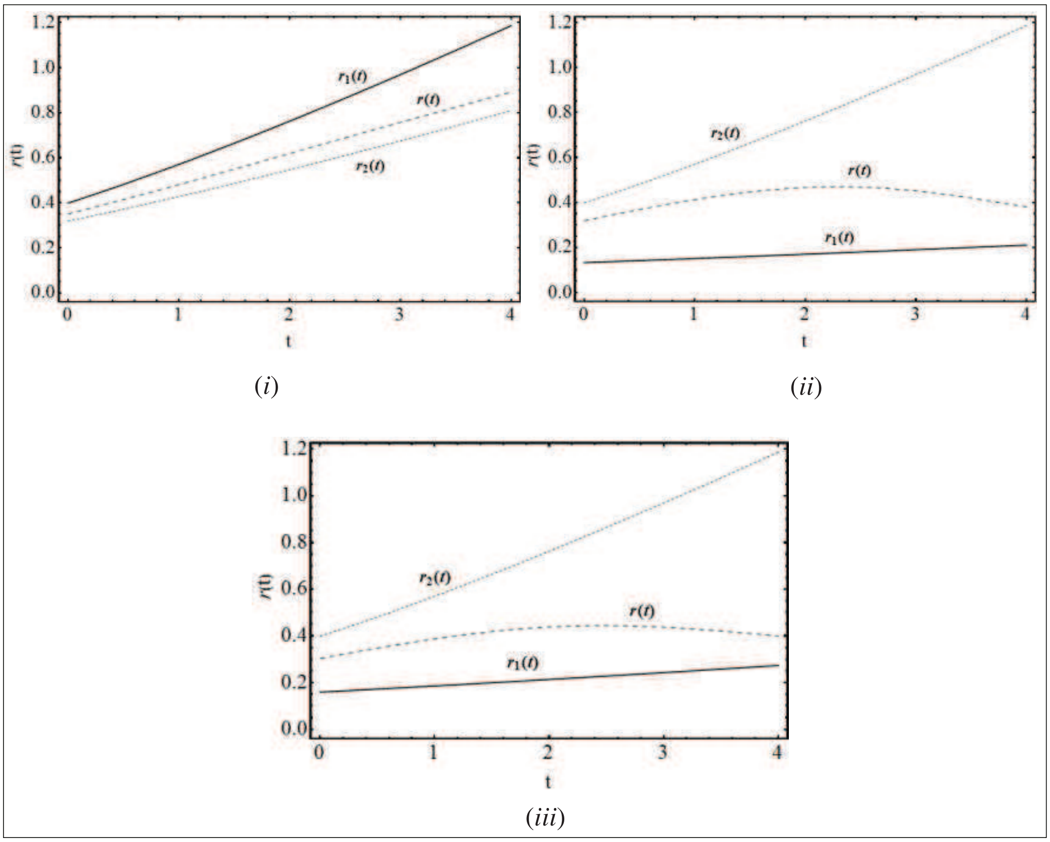

Figure 2: Hazard rate function components and their mixtures (*i*) (0.4, 2, 2.5); (*ii*) (0.3, 6, 2); (*iii*) (0.4, 5, 2)

The failure rate function of the two components mixture of half-normal distribution increases monotonically, concave down and approaches 0 as $t \to \infty$. The hazard rate function components and their mixtures are shown in Figure 2.

Mean, median and mode

The mean of the two components mixture of half-normal distributions is given by

$$\text{Mean} = \int_0^\infty t\left\{ p_1\sqrt{\frac{2}{\pi}}\lambda_1^{-1}\exp\left(\frac{-t^2}{2\lambda_1^2}\right) + p_2\sqrt{\frac{2}{\pi}}\lambda_2^{-1}\exp\left(\frac{-t^2}{2\lambda_2^2}\right)\right\}dt.$$

After simplifying this expression we obtain the following outcome; $\text{Mean} = \sqrt{\frac{2}{\pi}}\left(p_1\lambda_1 + p_2\lambda_2\right).$

The median and mode of the two components mixture of half-normal distributions are developed by solving the following equations with respect to t, respectively, i.e.,

$$p_1\left\{\operatorname{erf}\left(\frac{t}{\sqrt{2}\lambda_1}\right)\right\} + p_2\left\{erf\left(\frac{t}{\sqrt{2}\lambda_2}\right)\right\} = 0.5.$$

$$\frac{1}{\lambda_1^5\lambda_2^5}\exp\left\{\frac{-t^2\left(\lambda_1^2+\lambda_2^2\right)}{2\lambda_1^2\lambda_2^2}\right\}\sqrt{\frac{2}{\pi}}\left(\exp\left(\frac{t^2}{2\lambda_2^2}\right)p_1\left(t-\lambda_1\right)\left(t+\lambda_1\right)\lambda_2^5\right.$$

$$\left.+\exp\left(\frac{t^2}{2\lambda_1^2}\right)p_2\left(t-\lambda_2\right)\left(t+\lambda_2\right)\lambda_1^5\right) = 0.$$

The parametric values $(p_1, \lambda_1, \lambda_2)$ in Table 1 are chosen for the mixture density function with mixing proportion parameter p_1. The increasing and inverse behaviour has been noted for the mean, median, and mode when mixing proportion parameter p_1 increases. The standard half-normal distribution is discontinuous at $x = 0$ where the mode occurs. The mode of the mixture model is approximately 0 too.

Table 1: Mean, median and mode for the two-component mixture of half-normal distribution

$(p_1, \lambda_1, \lambda_2)$	Mean	Median	Mode	$(p_1, \lambda_1, \lambda_2)$	Mean	Median	Mode
0.2, 2, 3	2.15429	1.77624	2.3×10^{-9}	0.2, 5, 2	2.31387	1.68704	2.6×10^{-9}
0.4, 2, 3	2.07450	1.70281	4.6×10^{-9}	0.4, 5, 2	2.55323	1.83858	2.4×10^{-9}
0.5, 2, 3	1.99471	1.63383	7.6×10^{-9}	0.5, 2, 3	2.79260	2.01666	2.1×10^{-9}
0.6, 2, 3	1.91492	1.56914	1.2×10^{-9}	0.6, 5, 2	3.03196	2.22593	1.6×10^{-9}
0.7, 2, 3	1.83513	1.50855	1.7×10^{-9}	0.7 , 5, 2	3.27133	2.46976	1.1×10^{-9}

In real life executions, most of the times, it is not reasonable to proceed with the testing procedure until failure of the last object under testing. Censoring is a significant and effective aspect of lifetime applications. A worthy description on censoring is given in Gijbels (2010) and Kalbfleisch and Prentice (2011). In this article, an ordinary type-I right censoring is applied with a fixed life-test termination time for all objects.

Suppose n units from the above cited mixture model are used in the life testing experiment with a fixed test termination time T. After the test has been performed, it is observed that out of n units, r units have failed until the test termination time T, while the remaining (n-r) units are still working. Similar to the sampling scheme proposed by Mendenhall and Hader (1958), in many real life situations only the failure items can be identified as the members of the first and the second subpopulations, respectively. Out of r units, suppose r_1 and r_2 units belong to subpopulation-I and subpopulation-II, respectively. Here, it is clear that $r = (r_1 + r_2)$ and the remaining (n-r) units which are still functioning provide no information about the population to which they belong. Let x_{ij} be defined as the failure time of the j[th] unit from the i[th] subpopulation, where

$$j = 1, 2, ..., r_i,\ i = 1, 2, \text{ and } 0 < x_{1j}, x_{2j} \le T.$$

The likelihood function has the following form (cf. Everitt, 1985):

$$\mathrm{L}\left(\Delta \mid \mathbf{x}\right) \propto \left\{\prod_{j=1}^{r_1} p_1 f_1\left(x_{1j}\right)\right\}\left\{\prod_{j=1}^{r_2} p_2 f_2\left(x_{2j}\right)\right\}\left\{1 - F(T)\right\}^{n-r},$$

$$L\left(\Delta \mid \mathbf{x}\right) \propto \prod_{j=1}^{r_1}\left(p_1\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_1}\exp\left[-\frac{x_{1j}^{\ 2}}{2\lambda_1^{\ 2}}\right]\right)\prod_{j=1}^{r_2}\left(p_2\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_2}\exp\left[-\frac{x_{2j}^{\ 2}}{2\lambda_2^{\ 2}}\right]\right)\left\{1 - p_1\mathrm{erf}\left(\frac{T}{\sqrt{2}\lambda_1}\right) - p_2\mathrm{erf}\left(\frac{T}{\sqrt{2}\lambda_2}\right)\right\}^{(n-r)}, \quad ...(2)$$

$$L\left(\Delta \mid \mathbf{x}\right) \propto \sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{k_1} p_1^{\ r_1+k_1-k_2}\left(\lambda_1\right)^{-r_1}\exp\left\{-\frac{1}{\lambda_1^{\ 2}}\sum_{j=1}^{r_1}\left(\frac{x_{1j}^{\ 2}}{2}\right)\right\}\left\{\mathrm{erf}\left(\frac{T}{\sqrt{2}\lambda_1}\right)\right\}^{k_1-k_2}$$
$$\times\, p_2^{\ r_2+k_2}\ \left(\lambda_2\right)^{-r_2}\exp\left\{-\frac{1}{\lambda_1^{\ 2}}\sum_{j=1}^{r_2}\left(\frac{x_{2j}^{\ 2}}{2}\right)\right\}\left\{\mathrm{erf}\left(\frac{T}{\sqrt{2}\lambda_2}\right)\right\}^{k_2},$$

where $\Delta = \left(\lambda_1, \lambda_1, p_1\right)$.

Bayesian estimation of parameters

In this section, we discuss prior distributions for unknown parameters, loss functions and Bayes estimators with their posterior risks.

Bayesian estimation using informative prior

The Bayesian analysis requires the choice of suitable priors for the unknown parameters in addition to the experimental data. The main thing in this bond is the relationship between the prior distribution and the loss function. The mixture model under consideration has two scale parameters and one mixing proportion parameter. We considered both the informative and non-informative priors and observed the results. In keeping with Fernandez's (2000) assumptions, we consider that scale parameter $\lambda_i,\ i = 1,2$ has independent square-root inverted gamma priors with the shape and scale parameters as a_i, and b_i, respectively, $g(\lambda_i \mid a_i, b_i) \propto \lambda_i^{-(2a_i+1)} \exp(-b_i \lambda_i^{-2})$ and uniform prior for p_1. Combining the likelihood function given in equation (2) leads toward the following joint posterior distribution of λ_i, p_1 as:

$$g(\Delta \mid \mathrm{x}) = \frac{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{k_1} p_1^{r_1+k_1-k_2} p_2^{r_2+k_2} \prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}} \exp\left(-\left(\lambda_i^{-2}(\Lambda_i + b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}}{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{-k_1} B(r_1+k_1-k_2+1, r_2+k_2 \ +1)\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}} \exp\left(-\left(\lambda_i^{-2}(\Lambda_i + b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}\, d\lambda_1 d\lambda_2},$$

where $\Lambda_i = \sum_{i=1}^{r_i}\left(\frac{x_{ij}^{\ 2}}{2}\right), \Delta_1(\Phi_1) = \left\{ erf\left(\frac{T}{\sqrt{2}\lambda_1}\right)\right\}^{k_1-k}$, and $\Delta_2(\Phi_2) = \mathrm{e}\left\{ erf\left(\frac{T}{\sqrt{2}\lambda_2}\right)\right\}^{k_2}$, $i = 1,2.$

The marginal distribution of λ_1 is simply the probability distribution of λ_1 that neglects other nuisance information about λ_2 and p_1, which is obtained by integrating the joint probability distribution with respect to other parameters, thus,

$$g(\lambda_1 \mid \mathrm{x}) = \frac{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(\ldots)^{k_1} B(\ldots k \quad k \quad 1, r \quad k \quad 1)\lambda^{-\{r_1+(2a_1+1)\}} \exp(\ \ (\lambda^{-2}(\quad b)))\{\ \ (\ \ )\} \times\int_0^{\infty}\lambda_2^{-\{r_2+(2a_2+1)\}} \exp\left(-\left(\lambda_2^{-2}(\Lambda_2 + b_2)\right)\right)\{\Delta_2(\Phi_2)\}\, d\lambda_2}{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{-k_1} B(r_1+k_1-k_2+1, r_2+k_2 \ +1)\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}} \exp\left(-\left(\lambda_i^{-2}(\Lambda_i + b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}\, d\lambda_1 d\lambda_2},$$

Similarly, the marginal posterior distribution of λ_2 and p_1 are derived as:

$$g(\lambda_2 \mid \mathrm{x}) = \frac{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(\ldots)^{k_1} B(\ldots k \quad k \quad 1, r \quad k \quad 1)\lambda^{-\{r_2+(2a_2+1)\}} \exp(\ \ (\lambda^{-2}(\quad b)))\{\ \ (\ \ )\} \times\int_0^{\infty}\lambda_1^{-\{r_1+(2a_1+1)\}} \exp\left(-\left(\lambda_1^{-2}(\Lambda_1 + b_1)\right)\right)\{\Delta_1(\Phi_1)\}\, d\lambda_1}{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{-k_1} B(r_1+k_1-k_2+1, r_2+k_2 \ +1)\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}} \exp\left(-\left(\lambda_i^{-2}(\Lambda_i + b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}\, d\lambda_1 d\lambda_2},$$

and

$$g(p_1 \mid \mathrm{x}) = \frac{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{k_1} p_1^{r_1+k_1-k_2} p_2^{r_2+k_2} \int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}} \exp\left(-\left(\lambda_i^{-2}(\Lambda_i + b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}\, d\lambda_1 d\lambda_2}{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{-k_1} B(r_1+k_1-k_2+1, r_2+k_2 \ +1)\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}} \exp\left(-\left(\lambda_i^{-2}(\Lambda_i + b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}\, d\lambda_1 d\lambda_2}.$$

Bayesian estimation of the mixture model assuming the non-informative priors

The non-informative priors are a significant part of the Bayesian tool kit. The non-informative priors have the least effect on the ultimate inference comparative to the data. Bernardo & Smith (2000) contended that a non-informative prior should be considered as a reference prior, i.e., a prior that is favourable to use as a standard when scrutinising statistical data. The most common example of non-informative prior is uniform prior, which is employed when no conventional prior information is available.

Posterior distribution using uniform prior

A uniform prior for the unknown parameter λ_i can be written as $\lambda_i \sim Uniform(0,\infty),\ i=1,2.$ We suppose a prior that (λ_i, p_1) are independent and also assume that $p_1 \sim Uniform(0,1)$. Thus, the joint prior distribution of (λ_i, p_1) is $p(\lambda_i, p_1) \propto k.$ Merging the likelihood function given in equation (2) with a uniform prior information, we obtained the joint posterior distribution as:

$$g(\Delta \mid \mathrm{x}) = \frac{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{k_1} p_1^{r_1+k_1-k_2} p_2^{r_2+k_2} \prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}}\exp\left(-\left(\lambda_i^{-2}(\Lambda_i+b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}}{\sum_{k_1=0}^{n-r}\sum_{k_2=0}^{k_1}\binom{n-r}{k_1}\binom{k_1}{k_2}(-1)^{k_1} B(r_1+k_1-k_2+1, r_2+k_2+1)\int_0^\infty\int_0^\infty\prod_{i=1}^{2}\lambda_i^{-\{r_i+(2a_i+1)\}}\exp\left(-\left(\lambda_i^{-2}(\Lambda_i+b_i)\right)\right)\{\Delta_1(\Phi_1)\}\{\Delta_2(\Phi_2)\}\,d\lambda_1 d\lambda_2}.$$

Marginal distributions of λ_i and p_1 can be obtained by nuisance parameters. The expression for the marginal distributions under non-informative priors can be obtained with little modification.

Bayesian estimators under different loss functions

In order to select a best decision in decision theory, a suitable loss function must be specified. The preference of loss function is a difficult job, and its selection is often formed for the reasons of mathematical convenience without any particular decision problem of ongoing interest excluding the cost effect. As in the risk analysis, the potentiality of an undesired event and its consequences are both explored. This potentiality is usually measured by failure rate. The Bayesian approach is extensively applied to failure rate. In disastrous outcomes, it can be terrible to underestimate the potentiality of an event rather than to overestimate. This is significant when the risk level is the basis of risk reducing initiative, either by reducing the potentiality or the consequences. An inappropriately low estimate of the risk level can lead to the lack of necessary steps to reduce the risk level. Hence, it is unreasonable to use a loss function that allows the estimation of a failure probability of zero. A positive loss at the origin allows the estimation of zero and in risk analysis estimating a zero failure probability simply means that no risk is expected for further detail (Degroot, 1970). Six loss functions are used to obtain the Bayes estimators along with posterior risks, i.e., the squared error (*SE*) loss function, weighted squared error (*WSE*) loss function, the precautionary (*P*) loss function and quadratic (*Q*) loss function, modified squared error (*MSE*) loss function and squared-log error (*SLE*) loss function. The most commonly used loss function is *SE* loss function defined by $L_1 = \left(\hat{\theta}_{SE} - \theta\right)^2$, where $\hat{\theta}_{SE}$ is a decision rule to estimate parameter $\theta.$ Norstrom (1996) has introduced precautionary loss function. The *SLE* loss function, $L_6 = \left(\ln\hat{\theta} - \ln\theta\right)^2$, is a balanced loss function that takes both the error of estimation and goodness-of-fit into account but the unbalanced loss function only considers error of estimation. This loss function is convex for $\frac{\hat{\theta}}{\theta} \le e$ and concave otherwise, but its risk function has a unique minimum with respect to $\hat{\theta}$ (Dey, 2010). The Bayes estimators and posterior risks are given in Table 2.

Where *E* (in table 2) denotes the expectation with respect to the posterior distribution of $\theta.$ Thus, the posterior expectation of any function of parameter, say $U(\lambda_1, \lambda_2, p_1)$, can be written as:

$$\hat{U}(\lambda_1,\lambda_2,p_1) = E\{U(\lambda_1,\lambda_2,p_1) \mid (\mathrm{x})\} = \frac{\int_0^\infty\int_0^\infty\int_0^1 U(\lambda_1,\lambda_2,p_1)\, g(\lambda_1,\lambda_1,p_1 \mid \mathrm{x})\,dp_1 d\lambda_1 d\lambda_2}{\int_0^\infty\int_0^\infty\int_0^1 g(\lambda_1,\lambda_1,p_1 \mid \mathrm{x})\,dp_1 d\lambda_1 d\lambda_2}. \quad \ldots(3)$$

However, it is not possible to evaluate estimates of a set of parameters analytically. The estimates are not in the closed form and therefore, can be evaluated numerically. The use of numerical integration computer routines are required to converge the given data x in conjunction with the symbolic computational software Mathematica version 9.0 in order to evaluate the mixture density functions and determine the Bayes estimates and posterior risks.

Application

In this section, application of the half-normal mixture distribution, including the estimation of parameters and posterior risks for the comparison of loss functions and prior distributions for a given dataset, is presented. We choose a specific lifetime dataset because it shows positively skewed distribution.

Table 2: Bayes estimators and their posterior risks under different loss functions

Loss function	Bayes estimator	Posterior risk
$L_1=\left(\hat{\theta}_{SE}-\theta\right)^2$	$\hat{\theta}_{SE}=E(\theta\mid x)$	$\rho\left(\hat{\theta}_{SE}\right)=E\left(\theta^2\mid x\right)-\left\{E(\theta\mid x)\right\}^2$
$L_2=\theta^{-1}\left(\theta-\hat{\theta}_{WSE}\right)^2$	$\hat{\theta}_{WSE}=\left\{E\left(\theta^{-1}\mid x\right)\right\}^{-1}$	$\rho\left(\hat{\theta}_{WSE}\right)=E(\theta\mid x)-\left\{E\left(\theta^{-1}\mid x\right)\right\}^{-1}$
$L_3=\left(\theta-\hat{\theta}_p\right)^2\hat{\theta}_p^{-1}$	$\hat{\theta}_p=\sqrt{E\left(\theta^2\mid x\right)}$	$\rho\left(\hat{\theta}_p\right)=2\left\{\sqrt{E\left(\theta^2\mid x\right)}-E(\theta\mid x)\right\}$
$L_4=\left(1-\theta^{-1}\hat{\theta}_Q\right)^2$	$\hat{\theta}_Q=\left\{E\left(\theta^{-2}\mid x\right)\right\}^{-1}E\left(\theta^{-1}\mid x\right)$	$\rho\left(\hat{\theta}_Q\right)=1-\left\{E\left(\theta^{-2}\mid x\right)\right\}^{-1}\left\{E\left(\theta^{-1}\mid x\right)\right\}^2$
$L_5=\theta^{-2}\left(\theta-\hat{\theta}_{MSE}\right)^2$	$\hat{\theta}_{MSE}=\left\{E(\theta\mid x)\right\}^{-1}\left\{E\left(\theta^2\mid x\right)\right\}^{-1}$	$\rho\left(\hat{\theta}_{MSE}\right)=1-\left\{E(\theta\mid x)\right\}^{-1}\left\{E(\theta\mid x)\right\}^2$
$L_6=\left(\ln\hat{\theta}-\ln\theta\right)^2$	$\hat{\theta}_{SLE}=\exp\left\{E\left(\ln\theta\mid \mathbf{x}\right)\right\}$	$\rho\left(\hat{\theta}_{SLE}\right)=E\left\{(\ln\theta\mid \mathbf{x})\right\}^2-\left\{E\left(\ln\theta\mid \mathbf{x}\right)\right\}^2$

Table 3: Bayes estimates and their posterior risks in parentheses for real dataset

Loss functions	Uniform prior			Squared-root inverted gamma prior		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SE	1.18606	1.23481	0.41553	1.60764	1.20124	0.44055
	(0.12783)	(0.06798)	(0.00164)	(0.07508)	(0.03292)	(0.00410)
WSE	1.09125	1.17912	0.39798	1.56290	1.17523	0.43086
	(0.09480)	(0.05569)	(0.01755)	(0.04473)	(0.02601)	(0.00969)
P	1.23878	1.26204	0.41749	1.63082	1.21486	0.44519
	(0.10543)	(0.05445)	(0.00393)	(0.04639)	(0.02725)	(0.00927)
Q	0.99847	1.11118	0.39155	1.51992	1.15082	0.42077
	(0.08502)	(0.05708)	(0.01617)	(0.02751)	(0.02077)	(0.02343)
MSE	1.29384	1.28986	0.41947	1.65434	1.22864	0.44987
	(0.08329)	(0.04268)	(0.10938)	(0.02823)	(0.02230)	(0.02070)
SLE	1.13439	1.20652	0.40604	1.58499	1.18830	0.43173
	(0.08697)	(0.04675)	(0.03919)	(0.02823)	(0.02191)	(0.01983)

Description of the dataset

The dataset analysed by Leiva *et al*. (2009) corresponding to 72 survival times of guinea pigs injected with different doses of tubercle bacilli is utilised for this study. The coefficient of skewness is 0.80 and we used the half-normal mixture model to this data.

Now we assume that after failure occurs, we can identify the object by its cause for failure and categorise it as belonging to population I or population II. For computation ease, we have divided each observation by 100. It is assumed that the mixing weight and censoring rate are $p_1 = 0.4$ and $T = 2$, respectively. Using these assumptions, the observed data are divided into two censored subsamples of sizes $r_1 = 26$, and $r_2 = 37$, respectively. The following information is extracted for our mixture model:

$$n = 72, r = 63, \sum_{j=1}^{r_1} 0.5 x_{1j}^{\;2} = 8.14415 \text{ and}$$

$$\sum_{j=1}^{r_2} 0.5 x_{2j}^{\;2} = 12.9227.$$

Bayes estimates are obtained by assuming both priors using the informative and non-informative (uniform) priors under six loss functions. Bayes estimates and posterior risks for a real dataset are listed in Table 3. It is clear from these values that the optimal estimates are those with the minimum posterior risks and optimal estimates are obtained under mean squared error loss function. The results framework for evaluation of different loss functions is shown in Figure 3.

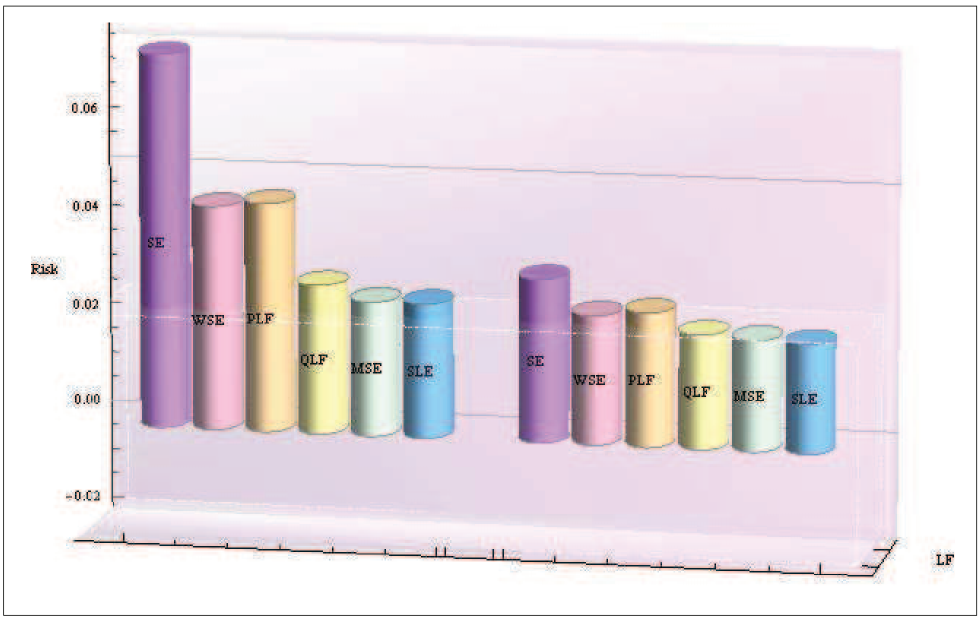

Figure 3: Bar diagram of estimates of two components mixture of half-normal distributions under different loss functions

RESULTS AND DISCUSSION

A simulation study illustrates the behaviour of the proposed estimators developed in previous sections for different sample sizes, different priors and different parametric values $(\lambda_1, \lambda_2) \in \{(1, 3), (4, 2)\}, T=6, p_1 \in 0.4$. Samples of sizes small, moderate and large i.e., n = 15, 30, 60 and 90, were generated from the two component mixture of half-normal distribution. A well-known procedure in simulation for computer generation of random variables is the inverse transform method. This method provides the most straightforward procedure to generate samples of a given distribution when its quantile function exists in closed-form. The Mathematica software includes the half-normal distribution in its software library. Probabilistic mixing is used to generate the mixture data. To generate the mixture model, a random number 'u' is generated from the uniform distribution (0, 1). If $u < p_1$, the observation is taken randomly from F_1 (the half-normal distribution with parameter λ_1), and if $u > p_1$ the observation is taken randomly from F_2 (the half-normal distribution with parameter λ_2). The values of hyperparameters (a_1, b_1, a_2, b_2) have been selected in such a manner that the prior mean becomes the approximate expected value of the corresponding parameter. The hyperparameters considered in the simulation study are {(2.5, 2, 2.5, 16), (1.5, 12, 2.5, 8)}. All observations that exceeded T were treated as censored. For each of the combinations of parameters and sample sizes, we generated 1000 samples using the Mathematica software. For each of 1000 samples, the average of these estimates and corresponding posterior risks are reported in Tables 4 and 5.

Table 4: Bayes estimates and their posterior risks in parentheses under different loss functions

$\Delta = (\lambda_1, \lambda_2, p_1)$		Uniform prior			Squared-root inverted gamma prior		
	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SE loss function	15	1.22905	3.67543	0.44263	1.25530	3.58802	0.42977
		(0.11873)	(0.39985)	(0.01914)	(0.06286)	(0.33696)	(0.01843)
	30	1.11755	3.22164	0.40642	0.98304	3.08622	0.40627
		(0.08109)	(0.30181)	(0.00732)	(0.03356)	(0.23482)	(0.00731)
(1, 3, 0.4)	60	1.05313	3.17528	0.40323	0.99122	3.02751	0.40323
		(0.02738)	(0.15464)	(0.00382)	(0.01922)	(0.12629)	(0.00380)
	90	1.02349	3.0667	0.40217	1.01784	2.9386	0.40217
		(0.01658)	(0.10179)	(0.00259)	(0.01229)	(0.08172)	(0.00259)
	15	3.69775	2.05261	0.36988	3.35641	2.08503	0.34136
		(0.04995)	(0.08976)	(0.06887)	(0.03955)	(0.03584)	(0.06401)
	30	3.62533	2.15081	0.39689	3.71823	1.95271	0.40341
		(0.45622)	(0.24573)	(0.00757)	(0.35652)	(0.11926)	(0.00741)
(4, 2, 0.4)	60	3.80876	2.08117	0.39978	3.90838	2.04883	0.39698
		(0.28636)	(0.09687)	(0.00392)	(0.25824)	(0.07827)	(0.00391)
	90	3.96582	2.05564	0.40133	3.91052	2.00667	0.39942
		(0.21755)	(0.06160)	(0.00264)	(0.18685)	(0.04504)	(0.00248)
WSE loss function	15	1.32998	3.44577	0.41201	1.25530	3.48902	0.42934
		(0.10873)	(0.29985)	(0.00936)	(0.06286)	(0.13696)	(0.00873)
	30	0.94173	3.18745	0.38718	0.93502	2.85961	0.38711
		(0.05046)	(0.09318)	(0.01916)	(0.02943)	(0.06749)	(0.01915)
(1, 3, 0.4)	60	1.21586	3.02658	0.39345	0.96041	2.94016	0.39346
		(0.02437)	(0.04777)	(0.00978)	(0.01909)	(0.03895)	(0.00963)
	90	1.02286	2.95938	0.39561	0.99033	3.04809	0.39561
		(0.01455)	(0.03089)	(0.00657)	(0.01211)	(0.02875)	(0.00657)
	15	3.43745	2.32961	0.39978	3.36695	2.38205	0.37186
		(0.04953)	(0.08776)	(0.06586)	(0.03681)	(0.03472)	(0.05891)
	30	3.48431	2.22763	0.3842	3.41229	1.98913	0.38183
		(0.11849)	(0.09487)	(0.02009)	(0.10543)	(0.05574)	(0.01975)
(4, 2, 0.4)	60	3.77855	2.05898	0.38823	3.70195	1.99681	0.39061
		(0.08387)	(0.04252)	(0.01022)	(0.07022)	(0.03038)	(0.01002)
	90	3.92207	2.02539	0.39255	3.83541	2.02214	0.39519
		(0.05638)	(0.02666)	(0.00678)	(0.05331)	(0.01818)	(0.00252)

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$\Delta = (\lambda_1, \lambda_2, p_1)$		Uniform prior			Squared-root inverted gamma prior		
	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
P loss function	15	1.39543	3.45324	0.42790	1.11642	3.35649	0.41897
		(0.10772)	(0.29531)	(0.00742)	(0.05299)	(0.12609)	(0.00711)
	30	1.21237	3.11833	0.41552	1.06259	3.10259	0.41516
		(0.07359)	(0.09281)	(0.01781)	(0.03294)	(0.07053)	(0.01779)
(1, 3, 0.4)	60	1.11069	3.09381	0.40795	1.03732	3.03182	0.40794
		(0.02643)	(0.04905)	(0.00942)	(0.01894)	(0.04114)	(0.00942)
	90	1.06344	3.02757	0.40537	0.99239	3.01764	0.40538
		(0.01568)	(0.03324)	(0.00643)	(0.01176)	(0.02923)	(0.00639)
	15	3.69775	2.05261	0.36988	3.35641	2.08503	0.34136
		(0.04995)	(0.08976)	(0.06887)	(0.03955)	(0.03584)	(0.06401)
	30	3.69593	2.43776	0.40258	3.65914	2.05261	0.41276
		(0.12945)	(0.10853)	(0.01913)	(0.09764)	(0.06318)	(0.01829)
(4, 2, 0.4)	60	3.77993	2.23003	0.40436	3.76182	2.03398	0.40542
		(0.08163)	(0.04852)	(0.00983)	(0.07051)	(0.03525)	(0.00968)
	90	4.03661	2.11949	0.40204	3.86459	2.01878	0.40201
		(0.06695)	(0.02579)	(0.00720)	(0.05467)	(0.02392)	(0.00664)

Table 5: Bayes estimates and their posterior risks in parentheses under different loss functions

$\Delta = (\lambda_1, \lambda_2, p_1)$		Uniform prior			Squared-root inverted gamma prior		
	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
Q loss function	15	1.48733	3.65031	0.49985	1.31980	3.49823	0.43842
		(0.23087)	(0.28098)	(0.00998)	(0.01951)	(0.19370)	(0.00784)
	30	0.98696	2.96186	0.36671	1.41763	2.87819	0.36705
		(0.04469)	(0.02802)	(0.05278)	(0.03029)	(0.02238)	(0.05279)
(1, 3, 0.4)	60	0.99158	3.06437	0.38334	1.30779	3.02741	0.38339
		(0.02168)	(0.01503)	(0.02569)	(0.01750)	(0.01308)	(0.02569)
	90	1.04047	3.01578	0.38889	1.19112	2.99097	0.38889
		(0.01351)	(0.01021)	(0.01698)	(0.01381)	(0.00905)	(0.01698)
	15	3.33628	2.45942	0.31979	3.26718	2.38804	0.35886
		(0.04748)	(0.07770)	(0.06406)	(0.03429)	(0.03340)	(0.03882)
	30	3.61375	2.05261	0.34941	3.32661	1.88513	0.36176
		(0.03982)	(0.03946)	(0.06081)	(0.03051)	(0.02537)	(0.05568)
(4, 2, 0.4)	60	3.86711	2.03619	0.37716	3.64096	1.99377	0.37951
		(0.01927)	(0.01805)	(0.02716)	(0.02088)	(0.01559)	(0.02712)
	90	3.90739	2.09226	0.38756	3.78116	1.99649	0.38594
		(0.01798)	(0.01637)	(0.00162)	(0.01434)	(0.01065)	(0.01693)

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$\Delta = (\lambda_1, \lambda_2, p_1)$		Uniform prior			Squared-root inverted gamma prior		
	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
MSE loss function	15	1.32235 (0.10642)	3.24556 (0.23268)	0.42983 (0.00692)	1.21180 (0.05983)	3.19853 (0.10654)	0.40345 (0.00432)
	30	1.14086 (0.05888)	3.33114 (0.03019)	0.42459 (0.04244)	0.97321 (0.03249)	3.08135 (0.02295)	0.42425 (0.04241)
(1, 3, 0.4)	60	1.05096 (0.02398)	3.10618 (0.01577)	0.41271 (0.02295)	0.98841 (0.01925)	2.98482 (0.01359)	0.41272 (0.02295)
	90	1.04379 (0.01382)	3.10182 (0.01048)	0.40863 (0.01573)	1.01012 (0.01297)	3.06015 (0.00941)	0.40863 (0.01573)
	15	3.37678 (0.04221)	2.65952 (0.05302)	0.30999 (0.04481)	3.56769 (0.03115)	2.28518 (0.03240)	0.39894 (0.05302)
	30	3.94799 (0.02815)	2.3129 (0.04251)	0.42074 (0.04415)	3.82711 (0.02969)	2.16284 (0.03382)	0.41751 (0.04659)
(4, 2, 0.4)	60	3.96136 (0.02109)	2.23552 (0.02104)	0.41855 (0.02424)	3.80782 (0.02075)	2.13286 (0.01786)	0.40639 (0.02466)
	90	3.98621 (0.01648)	2.14434 (0.01339)	0.40573 (0.01658)	3.60737 (0.08319)	2.08665 (0.04597)	0.39665 (0.09673)
SLE loss function	15	1.36544 (0.10877)	3.54290 (0.29872)	0.43229 (0.00943)	1.43358 (0.06036)	3.54281 (0.11792)	0.41432 (0.00769)
	30	0.98478 (0.05113)	2.87539 (0.03026)	0.38978 (0.04234)	0.87181 (0.03129)	2.91846 (0.02414)	0.38978 (0.04224)
(1, 3, 0.4)	60	1.15373 (0.02256)	2.90862 (0.01469)	0.39774 (0.02378)	1.09359 (0.01428)	2.96031 (0.01249)	0.39774 (0.02379)
	90	1.12839 (0.01397)	3.26685 (0.00961)	0.39959 (0.01468)	1.06704 (0.01148)	3.13452 (0.00218)	0.39960 (0.01425)
	15	3.37678 (0.04645)	2.65952 (0.06971)	0.30999 (0.06304)	3.56769 (0.03328)	2.28518 (0.03298)	0.36894 (0.05785)
	30	4.42596 (0.01860)	2.32052 (0.05203)	0.37663 (0.05036)	3.96137 (0.02299)	1.64128 (0.02524)	0.38876 (0.04289)
(4, 2, 0.4)	60	4.31887 (0.01599)	1.71492 (0.01721)	0.39648 (0.02417)	4.36991 (0.01088)	1.72367 (0.01384)	0.39684 (0.02403)
	90	4.33903 (0.01028)	1.87485 (0.0108)	0.39894 (0.01485)	4.10576 (0.01014)	1.98325 (0.01021)	0.39987 (0.01125)

From Tables 4 and 5, the foremost point that requires attention is that, those under the considerations of informative and non-informative prior beliefs, the estimated risks of Bayes estimators using different priors and loss functions reduce with an increase in sample size. In addition, the difference of the Bayes estimates from the assumed parameters approach zero with an increase in sample size. The behaviours of posterior risks have also been examined with the parametric values. For smaller parametric values, the posterior risks of Bayes estimators are smaller than the posterior risks of large parametric values considered in the simulation study. Bayesian estimates become very close to the true values of the parameters when sample size increases. Bayes estimates are found to be over - or underestimated in most of the cases under both priors. It is also observed that Bayes estimators are approximately equally efficient under mean squared error loss function and squared log error loss function. In fact, the use of these loss functions has unveiled the smallest posterior risk, which is really an advantageous property. As far as the problem of selecting a suitable prior is concerned, it can be seen that we obtained efficient results using the square-root gamma prior than the non-informative prior. Posterior risks for the Bayes estimates assuming uniform prior is also little high. The informative prior has a clear advantage over non-informative prior; therefore, we can easily make a selection of the preferable prior and loss function. It is worth mentioning that choice of the best prior and loss function is made on the posterior risks associated with it.

CONCLUSION

We have proposed a mixture of two-component half-normal distributions for a lifetime study and discussed the properties and estimation of parameters of the mixture distribution using six different loss functions under informative and non-informative priors. The capability of Bayesian analysis ensures this as a comprehensive study to address the selection of a suitable prior and desirable loss function for the mixture of half-normal distributions. It is seen that the closed form expression for the Bayes estimators are not possible, and we can only obtain the approximate Bayes estimates. The simulation study has revealed that an increase in sample size produced improved (concerning closeness) and reliable (concerning posterior risks) Bayes estimates. It is concluded that with an increase in sample size the posterior risks decrease. The posterior risks of the estimates of the set of the parameters appeared to be quite large with relatively large values of the parameters and *vice versa*. To estimate components as well as proportion parameters, priors can be ordered with respect to their performance as: informative prior < uniform prior. To address the problem of selecting prior and loss function, we have observed that the Bayes estimators of parameters perform best under mean *MSE* loss function assuming square-root gamma prior, and we can classify the risk under distinct loss functions in the following order: $MSELF \leq SLELF < QLF < WSELF < PLF < SELF$; and risk (informative) < risk (non-informative). The results drawn through real data coincides with simulated results.

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