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RESEARCH ARTICLE

Record values of ratio of Weibull random variables

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Abstract: In this paper we have derived the distribution of upper and lower record statistics for the ratio of two independently distributed Weibull random variables. The standard distributional properties of the resulting distribution have been studied. We have also obtained the recurrence relations for single and product moments of upper and lower record values for ratio of Weibull random variables. A real data application has also been given to see the performance of the proposed distribution.

Keywords: Lower record values, recurrence relations, upper record values, Weibull distribution.

INTRODUCTION

The Weibull distribution is widely used to model life time data. The distribution can deal with increasing, decreasing and constant failure-rates and can be created for data with and without suspensions (non-failures). Weibull distribution was first identified by Fréchet (1927) and first applied by Rosin and Rammler (1933) to describe a particle size distribution. The two parameters Weibull probability distribution function of a random variable 'W' is given by:

$$f(w)=\frac{\gamma}{\beta}w^{\gamma-1}\exp\left(-\frac{w^{\gamma}}{\beta}\right), x\geq 0, \beta\geq 0; \qquad ...(1)$$

where γ is the shape and β is the scale parameter of the distribution. The distribution function $F(w)$ and the hazard rate function corresponding to equation (1) are, respectively:

$$F(w)=1-\exp\left(-\frac{w^{\gamma}}{\beta}\right); \qquad ...(2)$$

and

$$h(w)=\frac{\gamma}{\beta}w^{\gamma-1}. \qquad ...(3)$$

The h^{th} moment for equation (1) is given as:

$$E\left(X^{h}\right)=\beta^{h/\gamma}\Gamma\left(1+h/\gamma\right). \qquad ...(4)$$

The mean and variance of the distribution can be obtained from equation (4).

Order and record statistics have emerged as important areas of study within the framework of ordered random variables. A comprehensive treaty of order statistics is given in David and Nagaraja (2003). Record statistics have been defined by Chandler (1952) and he has shown that the density function of k^{th} upper record statistics, when a random sample of size n is available from a distribution $F(x)$ is:

$$f\left(x_k\right)=\frac{1}{\Gamma(k)}f\left(x\right)\left[R\left(x\right)\right]^{k-1}; \qquad ...(5)$$

where $R\left(x\right)=-\ln\left[1-F\left(x\right)\right]$. Also, the distribution of lower records is

$$f_{L(n)}\left(x\right)=\frac{1}{\Gamma(n)}f\left(x\right)\left[H\left(x\right)\right]^{n-1}, \qquad ...(6)$$

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where $H(x) = -\ln[F(x)]$.

Ahsanullah (2004) has discussed record statistics for several distributions including exponential, Weibull and others. Specifically, the distribution of the k^{th} record for equation (1) is:

$$f(w_k) = \frac{\gamma}{\beta^k \Gamma(k)} w^{k\gamma-1} \exp\left(-\frac{w^\gamma}{\beta}\right). \quad \text{...(7)}$$

The h^{th} moment of the k^{th} record statistics of Weibull distribution is:

$$E\left(W_k^h\right) = \frac{\beta^{h/\gamma}}{\Gamma(k)} \Gamma\left(k + \frac{h}{\gamma}\right). \quad \text{...(8)}$$

In this paper the distribution of record statistics for ratio of Weibull random variables have been derived. Many authors have studied the order and record statistics in the context of specialised distributions. Shahbaz and Shahbaz (2012) have studied order statistics for Gumble distribution. Some other notable references are Shahbaz *et al*. (2009), and Shahbaz and Shahbaz (2009; 2010). The distribution of ratio of two Weibull random variables as discussed by Nadarajah (2010) has been given in the following section.

METHODOLOGY

The basic methodology contains distribution of ratio of two random variables as discussed by Nadarajah (2010), which plays a very important role in modelling various phenomenon, for example stress and strength study of components. In addition the methodology contains distributions of upper and lower records for ratio of Weibull random variables, which is given in following sections. Ali *et al.* (2007) have studied the distribution of ratio of two inverted Gamma variates and discussed some of its properties. Mekic *et al*. (2012) have studied the distribution of ratio of several random variables including Rayleigh distribution. The distribution of ratio of Weibull random variables has also been discussed by Mekic *et al.* (2012) and is given as below:

Let W_1 and W_2 be independently distributed Weibull random variables with common shape parameter γ and scale parameters β_1 and β_2. The probability density and distribution functions of W_1 and W_2 are, respectively:

$$f(w_1) = \frac{\gamma}{\beta_1} w_1^{\gamma-1} \exp\left(-\frac{w_1^\gamma}{\beta_1}\right); F(w_1) = 1 - \exp\left(-\frac{w_1^\gamma}{\beta_1}\right) \text{an}$$

$$f(w_2) = \frac{\gamma}{\beta_2} w_2^{\gamma-1} \exp\left(-\frac{w_2^\gamma}{\beta_2}\right); F(w_2) = \exp\left(-\frac{w_2^\gamma}{\beta_2}\right)$$

Let $X = W_1/W_2$ be the ratio of two Weibull random variables, then the density function of X can be obtained by using:

$$f_X(x) = \int_0^\infty w_2 f_{W_1}(xw_2) f_{W_2}(w_2) dw_2; \quad \text{...(9)}$$

and is given as

$$f(x) = \frac{\gamma\beta_1\beta_2 x^{\gamma-1}}{\left(\beta_1 + \beta_2 x^\gamma\right)^2}; x > 0. \quad \text{...(10)}$$

The distribution function of the ratio can be obtained by using:

$$F_X(x) = \int_0^\infty F_{W_1}(xw_2) f_{W_2}(w_2) dw_2. \quad \text{...(11)}$$

and is given as

$$F_X(x) = \frac{\beta_2 x^\gamma}{\beta_1 + \beta_2 x^\gamma}, x > 0. \quad \text{...(12)}$$

The h^{th} moment of ratio of Weibull variates is obtained as:

$$E\left(X^h\right) = \int_0^\infty x^h f(x) dx = \int_0^\infty \frac{\gamma\beta_1\beta_2 x^{h+\gamma-1}}{\left(\beta_1 + \beta_2 x^\gamma\right)^2} dx$$

$$= \left(\frac{h}{\gamma}\right)\left(\frac{\beta_1}{\beta_2}\right)^{h/\gamma} \Gamma\left(\frac{h}{\gamma}\right)\Gamma\left(1 - \frac{h}{\gamma}\right). \quad \text{...(13)}$$

The mean and variance of the ratio of Weibull variates can be obtained from equation (13). The values of mean, variance, skewness and kurtosis for distribution in equation (10) are given in Table I in Appendix. From the table we can see that the parameters β_1 and β_2 control the location and variability of the distribution, as for fixed γ change in these coefficients has no effect on skewness and kurtosis of the distribution. Further, we can see that as γ increases the mean, variance, skewness and kurtosis start decreasing. Also for fixed γ, increase in β_1 and β_2 bring increase in mean and variance of the distribution.

The distribution of upper record for Weibull variates has been derived in the following section.

RESULTS AND DISCUSSION

Shakil and Ahsanullah (2011) have studied the distributional properties of the upper records of ratio of Rayleigh variates. In this section we derive the distribution of upper as well as of lower records of ratio of Weibull variates. The density function of n^{th} upper record is given in equation (5) and that of lower record is given in equation (6).

Now using the density and distribution function of ratio of Weibull random variables in equation (5), the density of upper records is:

$$f_{U(n)}(x)=\frac{\gamma\beta_1\beta_2x^{\gamma-1}}{\Gamma(n)\left(\beta_1+\beta_2x^{\gamma}\right)^2}\left[\ln\left(\frac{\beta_1+\beta_2x^{\gamma}}{\beta_1}\right)\right]^{n-1},$$

$$n=1,2,3,.... \qquad ...(14)$$

The distribution function of n^{th} upper record for ratio of Weibull variates is readily written as

$$F_{U(n)}(x)=1-\frac{1}{\Gamma(n)}\Gamma\left(n,\log\left(\frac{\beta_2x^{\gamma}}{\beta_1}+1\right)\right). \qquad ...(15)$$

Again, using the density and distribution function of ratio of Weibull random variables in equation (6), the density function of n^{th} lower record is

$$f_{L(n)}(x)=\frac{\gamma\beta_1\beta_2x^{\gamma-1}}{\Gamma(n)\left(\beta_1+\beta_2x^{\gamma}\right)^2}\left[\ln\left(\frac{\beta_1}{\beta_2x^{\gamma}}+1\right)\right]^{n-1},$$

$$n=1,2,3,.... \qquad ...(16)$$

We now study some distributional properties of upper and lower records of ratio of Weibull variates in the following section. The distribution function of lower records is immediately written as

$$F_{L(n)}(x)=\frac{1}{\Gamma(n)}\Gamma\left(n,\log\left(\frac{\beta_1}{\beta_2x^{\gamma}}+1\right)\right) \qquad ...(17)$$

We now give distributional properties of upper and lower records in the following.

Distributional properties of records of ratio of Weibull variates

In this section, the moments, survival and hazard functions and entropy of the record values of ratio of Weibull random variables are obtained

Moments

The k^{th} moment of the n^{th} record value $X(n)$ is given as:

$$E\left[X^k_{U(n)}\right]=\int_{-\infty}^{\infty}x^k f\left(x_n\right)dx.$$

Using equation (14) in the above equation, we have:

$$E\left[X^k_{U(n)}\right]=\int_0^{\infty}x^k\frac{\gamma\beta_1\beta_2x^{\gamma-1}}{\Gamma(n)\left(\beta_1+\beta_2x^{\gamma}\right)^2}\left[\ln\left(\frac{\beta_1+\beta_2x^{\gamma}}{\beta_1}\right)\right]^{n-1}dx.$$

Making the transformation $t=\ln\left(\frac{\beta_1+\beta_2x^{\gamma}}{\beta_1}\right)$ in the above equation we have:

$$E\left[X^k_{U(n)}\right]=\frac{1}{\Gamma(n)}\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\int_0^{\infty}\left(e^t-1\right)^{k/\gamma}e^{-t}t^{n-1}dt$$

$$=\frac{1}{\Gamma(n)}\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\int_0^{\infty}\left(1-e^{-t}\right)^{k/\gamma}e^{-t(1-k/\gamma)}t^{n-1}dt.$$

Now using the following series expansion for positive real α :

$$\left(1+z\right)^{\alpha}=\sum_{h=o}^{\infty}\frac{\Gamma(\alpha+1)}{h!\Gamma(\alpha+1-h)}z^h;$$

we have

$$\left(1-e^{-t}\right)^{k/\gamma}=\sum_{h=0}^{\infty}\left(-1\right)^h\frac{\Gamma\left(k/\gamma+1\right)}{h!\Gamma\left(k/\gamma+1-h\right)}e^{-th}$$

and hence the k^{th} moment of the n^{th} record of the ratio of Weibull variates is:

$$E\left(X^k_{U(n)}\right)=\frac{1}{\Gamma(n)}\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\sum_{h=0}^{\infty}(-1)^h\frac{\Gamma(k/\gamma+1)}{h!\Gamma(k/\gamma+1-h)}$$

$$\int_0^{\infty}e^{-t(1-k/\gamma+h)}t^{n-1}dt$$

Now making the transformation $t\left(1-k/\gamma+h\right)=u$, the k^{th} moment of the n^{th} record value for ratio of Weibull random variables is:

$$E\left(X^k_{U(n)}\right)=\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\sum_{h=0}^{\infty}(-1)^h$$

$$\frac{\Gamma(k/\gamma+1)}{h!\Gamma(k/\gamma+1-h)}\left(\frac{\gamma}{\gamma+h\gamma-k}\right)^n. \qquad ...(18)$$

The mean and variance of the n^{th} upper record for ratio of two Weibull random variables can be obtained from equation (18). Table II contains values of mean, variance, skewness and kurtosis for distribution of record values of ratio of Weibull random variables. From Table II we can see that for fixed γ the mean and variance increase with increase in the value of n. The skewness and kurtosis first show a decrease and then increase with the increase in n. For fixed n the mean, variance, skewness and kurtosis decrease with increase in the value of γ. We can also see that for all values of n and γ, the distribution of upper records of ratio of Weibull variates is positively skewed.

Again, the k^{th} moment of lower record values for ratio of Weibull random variables is given as

$$E\left(X_{L(n)}^k\right)=\int_0^\infty x^k f_{L(n)}(x)\,dx=$$

$$\int_0^\infty x^k \frac{\gamma\beta_1\beta_2 x^{\gamma-1}}{\Gamma(n)\left(\beta_1+\beta_2 x^\gamma\right)^2}\left[\ln\left(\frac{\beta_1}{\beta_2 x^\gamma}+1\right)\right]^{n-1} dx$$

Making the transformation $\ln\left(\beta_1/\beta_2 x^\gamma+1\right)=t$ we have

$$E\left(X_{L(n)}^k\right)=\frac{1}{\Gamma(n)}\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\int_0^\infty\left(1-e^{-t}\right)^{-k/\gamma}e^{-t(1+k/\gamma)}t^{n-1}dx\,.$$

Now using the expansion

$$\left(1-x\right)^{-c}=\sum_{h=0}^\infty\frac{\Gamma(c+h)}{\Gamma(h)}x^h\ ;\ x<1,$$

we have

$$E\left(X_{L(n)}^k\right)=\frac{1}{\Gamma(n)}\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\sum_{h=0}^\infty\frac{\Gamma(k/\gamma+h)}{\Gamma(k/\gamma)}\int_0^\infty e^{-t(1+k/\gamma+h)}t^{n-1}dx$$

or

$$E\left(X_{L(n)}^k\right)=\left(\frac{\beta_1}{\beta_2}\right)^{k/\gamma}\sum_{h=0}^\infty\frac{\Gamma(k/\gamma+h)}{\Gamma(k/\gamma)}\left(\frac{\gamma}{\gamma h+\gamma+k}\right)^n \qquad ...(19)$$

We have computed the mean, variance, skewness and kurtosis of the distribution of lower records and are given in Table III for $\beta_1=\beta_2=1$.

From Table III we can see that for fixed γ, the mean and variance have a decreasing trend with increase in the value of n. The skewness shows an increasing trend with increase in n. Also, the kurtosis of lower records first shows a decreasing trend with increase in n and then shows an increasing trend with increase in n. Further, we can see that for fixed n, the mean of lower records shows increasing trend with increase in γ. The variance, skewness and kurtosis show a decreasing trend with increase in γ. The kurtosis shows an increasing trend only at $n = 2$. Further, we can see that the distribution of lower records is negatively skewed for all values of the parameters.

Survival and hazard functions

The survival and hazard functions of n^{th} record value X(n) with the pdf (14) and cdf (15) are respectively, given by

$$S_{U(n)}(x)=1-F_{U(n)}(x)=\frac{1}{\Gamma(n)}\Gamma\left(n,\log\left(\beta_2 x^\gamma/\beta_1+1\right)\right) \qquad ...(20)$$

The survivorship function for lower record is written from equations (16) and (17) as

$$S_{L(n)}(x)=1-F_{L(n)}(x)=1-\frac{1}{\Gamma(n)}\Gamma\left(n,\log\left(\beta_1/\beta_2 x^\gamma+1\right)\right). \qquad ...(21)$$

Graph of survival function of upper records for different choices of parameters β_1,β_2,γ and n are given in Figure I in Appendix.

The hazard rate function is given as

$$h_{U(n)}(x)=\frac{f_{U(n)}(x)}{1-F_{U(n)}(x)}=\frac{\beta_1\beta_2\gamma x^{\gamma-1}\log^{n-1}\left(\beta_2 x^\gamma/\beta_1+1\right)}{\left(\beta_1+\beta_2 x^\gamma\right)^2\Gamma\left(n,\log\left(\beta_2 x^\gamma/\beta_1+1\right)\right)}.$$

The hazard rate function for lower records is

$$h_{L(n)}(x)=\frac{f_{L(n)}(x)}{1-F_{L(n)}(x)}=\frac{\beta_1\beta_2\gamma x^{\gamma-1}\log^{n-1}\left(\beta_1/\beta_2 x^\gamma+1\right)}{\left(\beta_1+\beta_2 x^\gamma\right)^2\Gamma\left(n,\log\left(\beta_1/\beta_2 x^\gamma+1\right)\right)}.$$

The graph of hazard rate function of upper records for various values of parameters are given in Figure II in Appendix.

Entropy

The entropy of ratio of Weibull random variables is given as:

$$E\left[-\ln\left\{f(x)\right\}\right]=\int_0^\infty-\ln\left[\frac{\gamma\beta_1\beta_2 x^{\gamma-1}}{\left(\beta_1+\beta_2 x^\gamma\right)^2}\right]\frac{\gamma\beta_1\beta_2 x^{\gamma-1}}{\left(\beta_1+\beta_2 x^\gamma\right)^2}dx$$

which on simplification becomes:

$$E\left[-\ln\left\{f\left(x\right)\right\}\right]=2-\frac{1}{\gamma}\ln\left(\frac{\beta_2}{\beta_1}\right)-\ln\left(\gamma\right) \quad ...(22)$$

The plot of entropy can be made from equation (22).

Estimation using records

In this section we have given estimation and application of the distribution of ratio of Weibull random variables. The estimation and application are given based upon the records.

Estimation using upper records

The density and distribution function of the ratio of Weibull random variables are given in equations (10) and (12) as

$$f\left(x\right)=\frac{\gamma\beta_1\beta_2x^{\gamma-1}}{\left(\beta_1+\beta_2x^{\gamma}\right)^2}. \quad ...(10)$$

and

$$F_X\left(x\right)=\frac{\beta_2x^{\gamma}}{\beta_1+\beta_2x^{\gamma}}. \quad ...(12)$$

We now give estimation of parameters of the distribution of records. The joint distribution of records is given by Shahbaz *et al*. (2016) as

$$f_{1,2,\ldots,n}\left(x_1,x_2,\ldots,x_n\right)=\left[\prod_{i=1}^{n-1}r\left(x_i\right)\right]f\left(x_n\right), \quad ...(23)$$

where $r\left(x_i\right)=f\left(x_i\right)/\left\{1-F\left(x_i\right)\right\}$. Now using equations (10) and (12) we have

$$r\left(x_i\right)=\frac{\gamma\beta_2x^{\gamma-1}}{\beta_1+\beta_2x^{\gamma}}.$$

The joint distribution of records for ratio of Weibull random variables is therefore

$$L\left(\mathbf{x}\,|\,\beta_1,\beta_2,\gamma\right)=f_{1,2,\ldots,n}\left(x_1,x_2,\ldots,x_n\right)$$

$$=\left(\prod_{i=1}^{n-1}\frac{\gamma\beta_2x_i^{\gamma-1}}{\beta_1+\beta_2x_i^{\gamma}}\right)\times\frac{\gamma\beta_1\beta_2x_n^{\gamma-1}}{\left(\beta_1+\beta_2x_n^{\gamma}\right)^2}$$

$$=\frac{\gamma^n\beta_1\beta_2^n\prod_{i=1}^{n}x_i^{\gamma-1}}{\prod_{i=1}^{n-1}\left(\beta_1+\beta_2x_i^{\gamma}\right)}\times\frac{1}{\left(\beta_1+\beta_2x_n^{\gamma}\right)^2}.$$

The log of likelihood function is

$$\ln L\left(\mathbf{x}\,|\,\beta_1,\beta_2,\gamma\right)=n\ln\gamma+\ln\beta_1+n\ln\beta_2+\left(\gamma-1\right)$$

$$\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n-1}\ln\left(\beta_1+\beta_2x_i^{\gamma}\right)-2\ln\left(\beta_1+\beta_2x_n^{\gamma}\right). \quad ...(24)$$

The derivatives of likelihood function with respect to the parameters are

$$\frac{\partial\ln LF\left(\mathbf{x}\right)}{\partial\beta_1}=\frac{1}{\beta_1}-\frac{2}{\beta_1+\beta_2x_n^{\gamma}}-\sum_{i=1}^{n-1}\frac{1}{\beta_1+\beta_2x_i^{\gamma}} \quad ...(26)$$

$$\frac{\partial\ln LF\left(\mathbf{x}\right)}{\partial\beta_2}=\frac{n}{\beta_2}-\frac{2x_n^{\gamma}}{\beta_1+\beta_2x_n^{\gamma}}-\sum_{i=1}^{n-1}\frac{x_i^{\gamma}}{\beta_1+\beta_2x_i^{\gamma}} \quad ...(27)$$

$$\frac{\partial\ln LF\left(\mathbf{x}\right)}{\partial\gamma}=\frac{n}{\gamma}-\frac{\beta_2x_n^{\gamma}\ln x_n}{\beta_1+\beta_2x_n^{\gamma}}+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\frac{\beta_2x_i^{\gamma}\ln x_i}{\beta_1+\beta_2x_i^{\gamma}} \quad ...(28)$$

The maximum likelihood estimates can be obtained by simultaneously solving the equations

$$\frac{\partial\ln LF\left(\mathbf{x}\right)}{\partial\beta_1}=0;\ \frac{\partial\ln LF\left(\mathbf{x}\right)}{\partial\beta_2}=0\ \text{ and }\ \frac{\partial\ln LF\left(\mathbf{x}\right)}{\partial\gamma}=0.$$

The entries of Fisher information matrix are given below

$$\frac{\partial^2\ln LF\left(\mathbf{x}\right)}{\partial\beta_1^2}=-\frac{1}{\beta_1^2}+\frac{2}{\left(\beta_1+\beta_2x_n^{\gamma}\right)^2}+\sum_{i=1}^{n-1}\frac{1}{\left(\beta_1+\beta_2x_i^{\gamma}\right)^2}$$

$$\frac{\partial^2\ln LF\left(\mathbf{x}\right)}{\partial\beta_2^2}=-\frac{n}{\beta_2^2}+\frac{2x_n^{2\gamma}}{\left(\beta_1+\beta_2x_n^{\gamma}\right)^2}+\sum_{i=1}^{n-1}\frac{x_i^{2\gamma}}{\left(\beta_1+\beta_2x_i^{\gamma}\right)^2}$$

$$\frac{\partial^2\ln LF\left(\mathbf{x}\right)}{\partial\gamma^2}=-\frac{n}{\gamma^2}-\frac{2\beta_1\beta_2x_n^{\gamma}\left(\ln x_n\right)^2}{\left(\beta_1+\beta_2x_n^{\gamma}\right)^2}+\sum_{i=1}^{n-1}\frac{\beta_2^2x_i^{2\gamma}\left(\ln x_i\right)^2}{\left(\beta_1+\beta_2x_i^{\gamma}\right)^2}$$

$$-\sum_{i=1}^{n-1}\frac{\beta_2x_i^{\gamma}\left(\ln x_i\right)^2}{\beta_1+\beta_2x_i^{\gamma}}$$

$$\frac{\partial^2\ln LF\left(\mathbf{x}\right)}{\partial\beta_1\partial\beta_2}=\frac{2x_n^{\gamma}}{\left(\beta_1+\beta_2x_n^{\gamma}\right)^2}-\sum_{i=1}^{n-1}\frac{x_i^{\gamma}}{\left(\beta_1+\beta_2x_i^{\gamma}\right)^2}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_1\partial\gamma} = \frac{2\beta_2 x_n^\gamma \ln x_n}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{\beta_2 x_i^\gamma \ln x_i}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_2\partial\gamma} = -\frac{2\beta_1 x_n^\gamma \ln x_n}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{\beta_2 x_i^{2\gamma} \ln x_i}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2} - \sum_{i=1}^{n-1}\frac{x_i^\gamma \ln x_i}{\beta_1+\beta_2 x_i^\gamma}$$

The entries of Fisher information matrix can be computed for any given data.

Estimation using lower records

In this subsection, we have discussed the estimation of parameters based upon lower records. The joint distribution of lower records is given by Shahbaz *et al*. (2016) as

$$f_{1,2,\dots,n}\left(x_1,x_2,\dots,x_n\right) = \left[\prod_{i=1}^{n-1}\frac{f\left(x_i\right)}{F\left(x_i\right)}\right] f\left(x_n\right). \qquad ...(28)$$

Now using the distribution of ratio of Weibull variates from equations (10) and (12), in equation (28) we have

$$L(\mathbf{x}\,|\,\beta_1,\beta_2,\gamma) = f_{1,2,\dots,n}\left(x_1,x_2,\dots,x_n\right)$$

$$= \left(\prod_{i=1}^{n-1}\frac{\gamma\beta_1 x_i^{-1}}{\beta_1+\beta_2 x_i^\gamma}\right)\times\frac{\gamma\beta_1\beta_2 x_n^{\gamma-1}}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2}$$

$$= \frac{\gamma^n \beta_1^n \beta_2 \prod_{i=1}^{n} x_i^{-1}}{\prod_{i=1}^{n-1}\left(\beta_1+\beta_2 x_i^\gamma\right)}\times\frac{1}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2}\cdot$$

The log of likelihood function is

$$\ln L(\mathbf{x}\,|\,\beta_1,\beta_2,\gamma) = n\ln\gamma + n\ln\beta_1 + \ln\beta_2 - \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n-1}\ln\left(\beta_1+\beta_2 x_i^\gamma\right) - 2\ln\left(\beta_1+\beta_2 x_n^\gamma\right)\cdot \qquad ...(29)$$

Now the derivatives of log of likelihood function with respect to the parameters are given below

$$\frac{\partial \ln LF(\mathbf{x})}{\partial\beta_1} = \frac{n}{\beta_1} - \frac{2}{\beta_1+\beta_2 x_n^\gamma} - \sum_{i=1}^{n-1}\frac{1}{\beta_1+\beta_2 x_i^\gamma} \qquad ...(30)$$

$$\frac{\partial \ln LF(\mathbf{x})}{\partial\beta_2} = \frac{1}{\beta_2} - \frac{2x_n^\gamma}{\beta_1+\beta_2 x_n^\gamma} - \sum_{i=1}^{n-1}\frac{x_i^\gamma}{\beta_1+\beta_2 x_i^\gamma} \qquad ...(31)$$

$$\frac{\partial \ln LF(\mathbf{x})}{\partial\gamma} = \frac{n}{\gamma} - \frac{2\beta_2 x_n^\gamma \ln x_n}{\beta_1+\beta_2 x_n^\gamma} - \sum_{i=1}^{n-1}\frac{\beta_2 x_i^\gamma \ln x_i}{\beta_1+\beta_2 x_i^\gamma} \qquad ...(32)$$

The maximum likelihood estimates can be obtained by simultaneously solving the equations

$$\frac{\partial \ln LF(\mathbf{x})}{\partial\beta_1} = 0;\ \frac{\partial \ln LF(\mathbf{x})}{\partial\beta_2} = 0 \text{ and } \frac{\partial \ln LF(\mathbf{x})}{\partial\gamma} = 0.$$

The entries of Fisher information matrix are given below

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_1^2} = -\frac{n}{\beta_1^2} + \frac{2}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{1}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_2^2} = -\frac{1}{\beta_2^2} + \frac{2x_n^{2\gamma}}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{x_i^{2\gamma}}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\gamma^2} = -\frac{n}{\gamma^2} - \frac{2\beta_1\beta_2 x_n^\gamma \left(\ln x_n\right)^2}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{\beta_2^2 x_i^{2\gamma}\left(\ln x_i\right)^2}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2} - \sum_{i=1}^{n-1}\frac{\beta_2 x_i^\gamma \left(\ln x_i\right)^2}{\beta_1+\beta_2 x_i^\gamma}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_1\partial\beta_2} = \frac{2x_n^\gamma}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{x_i^\gamma}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_1\partial\gamma} = \frac{2\beta_2 x_n^\gamma \ln x_n}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} - \sum_{i=1}^{n-1}\frac{\beta_2 x_i^\gamma \ln x_i}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2}$$

$$\frac{\partial^2 \ln LF(\mathbf{x})}{\partial\beta_2\partial\gamma} = -\frac{2\beta_1 x_n^\gamma \ln x_n}{\left(\beta_1+\beta_2 x_n^\gamma\right)^2} + \sum_{i=1}^{n-1}\frac{\beta_2 x_i^{2\gamma} \ln x_i}{\left(\beta_1+\beta_2 x_i^\gamma\right)^2} - \sum_{i=1}^{n-1}\frac{x_i^\gamma \ln x_i}{\beta_1+\beta_2 x_i^\gamma}\cdot$$

The entries of Fisher information matrix can be computed for given data.

Application

In this sub-section we have given the application of the distribution of ratio of Weibull random variables. The data we have used is exceedness of flood peaks (in m^3s^{-1}) of Wheaton river as used by Choulakian and Stephens (2001). The data is given below

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7	0.1	1.1	0.6
9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0

Table 1: Fitted distribution values

	Parameters (standard error in parenthesis)			AIC	A-D Test (p - value)
Weibull	$\lambda = 11.630^{***}$ (1.602)	$\gamma = 0.901^{***}$ (0.086)		506.997	0.844 (0.5501)
Exponentiated Weibull	$\lambda = 0.019^{*}$ (0.029)	$\gamma = 1.346^{**}$ (0.401)	$\alpha = 0.541^{*}$ (0.234)	508.055	57.510 (< 0.0001)
Kumaraswamy Weibull	$\alpha = 1.198^{***}$ (0.383)	$b = 0.127^{***}$ (0.016)	$\gamma = 0.857^{***}$ (0.006)	508.831	62.489 (< 0.0001)
Ratio of Weibull	$\beta = 2.422^{**}$ (0.683)	$\beta_2 = 0.238^{**}$ (0.067)	$\gamma = 1.212^{***}$ (0.011)	502.678	1.472 (0.1833)
Record values of ratio of Weibull	$\beta = 1.922^{**}$ (0.483)	$\beta_2 = 0.174^{***}$ (0.032)	$\gamma = 1.412^{***}$ (0.019)	497.226	0.625 (0.6241)

***: p-value < 0.0001; **: p-value < 0.001; *: p-value < 0.05; x: p-value > 0.05

We have fitted the distribution of upper records of ratio of Weibull random variables, ratio of Weibull distribution, exponentiated Weibull, Kumaraswamy Weibull and Weibull distribution to above data. The parameter estimates are given in Table 1.

From Table 1 we can see that the distribution of record values of ratio of two Weibull variables fits the data well as compared with competing distributions.

Recurrence relations for moments of record values

In this section, we have given the recurrence relations for moments of upper and lower records for the distribution of ratio of Weibull random variables. These recurrence relations are derived in the following sub-sections.

Recurrence relations for moments of lower record values

In this sub-section we have derived the recurrence relations for moments of lower records of ratio of Weibull random variables. For this, we first see whether the density and distribution function of ratio of Weibull random variables are related as

$$F(x) = \left[\frac{x}{\gamma} + \frac{\beta_2}{\beta_1 \gamma} x^{\gamma+1}\right] f(x). \qquad \text{...(33)}$$

We also have the following relations between moments of lower record values as given by Ahsanullah (2004)

$$\mu_{L(n)}^{p}-\mu_{L(n-1)}^{p}=-\frac{p}{\Gamma(n)}\int_{-\infty}^{\infty}x^{p-1}F(x)\left[H(x)\right]^{n-1}dx \quad ...(34)$$

and

$$\mu_{L(m,n)}^{p,q}-\mu_{L(m-1,n)}^{p,q}=-\frac{q}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^q \times\frac{f(x_1)}{F(x_1)}\left[H(x_1)\right]^{m-1}\times\left[H(x_2)-H(x_1)\right]^{n-m-1} F(x_2)\,dx_2 dx_1. \quad ...(35)$$

Now using equation (33) in equation (34), we have the following recurrence relation between single moments of lower record values for ratio of Weibull random variables

$$\mu_{L(n)}^{p}-\mu_{L(n-1)}^{p}=-\frac{p}{\Gamma(n)}\int_{0}^{\infty}x^{p-1}\left[\left\{\frac{x}{\gamma}+\frac{\beta_2}{\beta_1\gamma}x^{\gamma+1}\right\}f(x)\right]\left[H(x)\right]^{n-1}dx=-\frac{p}{\gamma}\left[\mu_{L(n)}^{p}+\frac{\beta_2}{\beta_1}\mu_{L(n)}^{p+\gamma}\right]$$ or

$$\mu_{L(n)}^{p}=\frac{\gamma}{p+\gamma}\left[\mu_{L(n-1)}^{p}-\frac{p\beta_2}{\gamma\beta_1}\mu_{L(n)}^{p+\gamma}\right]. \quad ...(36)$$

Again using equation (33) in equation (35), we have the following recurrence relation between product moments of lower record values for ratio of Weibull random variables

$$\mu_{L(m,n)}^{p,q}-\mu_{L(m-1,n)}^{p,q}=-\frac{q}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-1}\times\frac{f(x_1)}{F(x_1)}\left[H(x_1)\right]^{m-1}\times\left[H(x_2)-H(x_1)\right]^{n-m-1}\left[\frac{x_2}{\gamma}+\frac{\beta_2}{\beta_1\gamma}x_2^{\gamma+1}\right]f(x_2)\,dx_2 dx_1$$

$$=-\frac{q}{\gamma}\left(\mu_{L(m,n)}^{p,q}+\frac{\beta_2}{\beta_1}\mu_{L(m,n)}^{p,q+\gamma}\right)$$ or

$$\mu_{L(m,n)}^{p,q}=\frac{\gamma}{\gamma+q}\left[\mu_{L(m-1,n)}^{p,q}-\frac{q\beta_2}{\gamma\beta_1}\mu_{L(m,n)}^{p,q+\gamma}\right]. \quad ...(37)$$

The single and product moments for lower records can be computed recursively from equations (36) and (37).

Recurrence relations for moments of upper record values

In this sub-section we have derived the recurrence relations for moments of upper records of ratio of Weibull random variables. For this, first we see whether the density and survivorship function of ratio of Weibull random variables are related as

$$\bar{F}(x)=\left[\frac{x}{\gamma}+\frac{\beta_1}{\beta_2\gamma}x^{1-\gamma}\right]f(x). \quad ...(38)$$

We also have the following relations between moments of lower record values as given by Ahsanullah (2004)

$$\mu_{U(n)}^{p}-\mu_{U(n-1)}^{p}=\frac{p}{\Gamma(n)}\int_{-\infty}^{\infty}x^{p-1}\left[1-F(x)\right]\left[R(x)\right]^{n-1}dx \quad ...(39)$$

and

$$\mu_{U(m,n)}^{p,q}-\mu_{U(m-1,n)}^{p,q}=\frac{q}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^q \times r(x_1)\left[R(x_1)\right]^{m-1}\times\left[R(x_2)-R(x_1)\right]^{n-m-1}\left[1-F(x_2)\right]dx_2 dx_1. \quad ...(40)$$

Now using equation (38) in equation (39), the following recurrence relation is obtained for single moments of upper record values for ratio of Weibull random variables

$$\mu_{U(n)}^{p}-\mu_{U(n-1)}^{p}=\frac{p}{\Gamma(n)}\int_{0}^{\infty}x^{p-1}\left[\left\{\frac{x}{\gamma}+\frac{\beta_1}{\beta_2\gamma}x^{1-\gamma}\right\}f(x)\right]\left[R(x)\right]^{n-1}dx=\frac{p}{\gamma}\left[\mu_{U(n)}^{p}+\frac{\beta_1}{\beta_2}\mu_{U(n)}^{p-\gamma}\right]$$

or

$$\mu_{U(n)}^{p}=\frac{\gamma}{p-\gamma}\left[\mu_{U(n-1)}^{p}+\frac{p\beta_1}{\gamma\beta_2}\mu_{U(n)}^{p-\gamma}\right]. \quad ...(41)$$

Again using equation (38) in equation (40), we have the following recurrence relation between product moments of lower record values for ratio of Weibull random variables

$$\mu_{m,n}^{p,q}-\mu_{m-1,n}^{p,q}=\frac{q}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-1}$$

$$\times r(x_1)\left[R(x_1)\right]^{m-1}\times\left[R(x_2)-R(x_1)\right]^{n-m-1}$$

$$\left[\frac{x_2}{\gamma}+\frac{\beta_1}{\beta_2\gamma}x_2^{1-\gamma}\right]f(x_2)\,dx_2dx_1$$

$$=\frac{q}{\gamma}\left(\mu_{U(m,n)}^{p,q}+\frac{\beta_1}{\beta_2}\mu_{U(m,n)}^{p,q-\gamma}\right)$$

or

$$\mu_{U(m,n)}^{p,q}=\frac{\gamma}{\gamma-q}\left[\mu_{U(m-1,n)}^{p,q}+\frac{q\beta_1}{\gamma\beta_2}\mu_{m,n}^{p,q-\gamma}\right]. \quad ...(42)$$

The single and product moments for lower records can be computed recursively from equations (41) and (42).

CONCLUSION

The distribution of upper records has simple density function, whereas the distribution of lower records has a slightly complex density. The distribution of upper records for ratio of Weibull random variables is positively skewed, whereas the distribution of lower records is negatively skewed. The moments of records values for ratio of Weibull variates are nicely related and hence the higher order moments can be computed from the lower order moments.

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Appendix

Table I: Mean, variance (Var), skewness (Sk) and kurtosis (Ku) of the distribution

		γ											
		γ = 4.5				γ = 5.5				γ = 6.5			
β_1	β_2	Mean	Var	Sk	Ku	Mean	Var	Sk	Ku	Mean	Var	Sk	Ku
1.5	1.5	1.09	0.24	0.11	62.02	1.06	0.14	- 0.91	19.50	1.04	0.09	- 1.38	12.07
	2.0	1.16	0.27	0.11	62.02	1.11	0.16	- 0.91	19.50	1.09	0.10	- 1.38	12.07
	2.5	1.22	0.30	0.11	62.02	1.16	0.17	- 0.91	19.50	1.13	0.11	- 1.38	12.07
	3.0	1.27	0.32	0.11	62.02	1.20	0.18	- 0.91	19.50	1.16	0.12	- 1.38	12.07
	3.5	1.31	0.35	0.11	62.02	1.23	0.19	- 0.91	19.50	1.18	0.12	- 1.38	12.07
	4.0	1.35	0.37	0.11	62.02	1.26	0.20	- 0.91	19.50	1.21	0.13	- 1.38	12.07
2.0	1.5	1.02	0.21	0.11	62.02	1.00	0.13	- 0.91	19.50	0.99	0.09	- 1.38	12.07
	2.0	1.09	0.24	0.11	62.02	1.06	0.14	- 0.91	19.50	1.04	0.09	- 1.38	12.07
	2.5	1.14	0.26	0.11	62.02	1.10	0.15	- 0.91	19.50	1.08	0.10	- 1.38	12.07
	3.0	1.19	0.29	0.11	62.02	1.14	0.16	- 0.91	19.50	1.11	0.11	- 1.38	12.07
	3.5	1.23	0.31	0.11	62.02	1.17	0.17	- 0.91	19.50	1.13	0.11	- 1.38	12.07
	4.0	1.27	0.32	0.11	62.02	1.20	0.18	- 0.91	19.50	1.16	0.12	- 1.38	12.07
2.5	1.5	0.97	0.19	0.11	62.02	0.96	0.12	- 0.91	19.50	0.96	0.08	- 1.38	12.07
	2.0	1.03	0.22	0.11	62.02	1.01	0.13	- 0.91	19.50	1.00	0.09	- 1.38	12.07
	2.5	1.09	0.24	0.11	62.02	1.06	0.14	- 0.91	19.50	1.04	0.09	- 1.38	12.07
	3.0	1.13	0.26	0.11	62.02	1.09	0.15	- 0.91	19.50	1.07	0.10	- 1.38	12.07
	3.5	1.17	0.28	0.11	62.02	1.12	0.16	- 0.91	19.50	1.10	0.10	- 1.38	12.07
	4.0	1.21	0.29	0.11	62.02	1.15	0.17	- 0.91	19.50	1.12	0.11	- 1.38	12.07
3.0	1.5	0.93	0.18	0.11	62.02	0.93	0.11	- 0.91	19.50	0.93	0.08	- 1.38	12.07
	2.0	0.99	0.20	0.11	62.02	0.98	0.12	- 0.91	19.50	0.98	0.08	- 1.38	12.07
	2.5	1.04	0.22	0.11	62.02	1.02	0.13	- 0.91	19.50	1.01	0.09	- 1.38	12.07
	3.0	1.09	0.24	0.11	62.02	1.06	0.14	- 0.91	19.50	1.04	0.09	- 1.38	12.07
	3.5	1.12	0.26	0.11	62.02	1.09	0.15	- 0.91	19.50	1.06	0.10	- 1.38	12.07
	4.0	1.16	0.27	0.11	62.02	1.11	0.16	- 0.91	19.50	1.09	0.10	- 1.38	12.07
3.5	1.5	0.90	0.16	0.11	62.02	0.91	0.10	- 0.91	19.50	0.91	0.07	- 1.38	12.07
	2.0	0.96	0.19	0.11	62.02	0.95	0.11	- 0.91	19.50	0.95	0.08	- 1.38	12.07
	2.5	1.01	0.21	0.11	62.02	0.99	0.12	- 0.91	19.50	0.99	0.08	- 1.38	12.07
	3.0	1.05	0.22	0.11	62.02	1.03	0.13	- 0.91	19.50	1.02	0.09	- 1.38	12.07
	3.5	1.09	0.24	0.11	62.02	1.06	0.14	- 0.91	19.50	1.04	0.09	- 1.38	12.07
	4.0	1.12	0.25	0.11	62.02	1.08	0.15	- 0.91	19.50	1.06	0.10	- 1.38	12.07
4.0	1.5	0.87	0.15	0.11	62.02	0.88	0.10	- 0.91	19.50	0.89	0.07	- 1.38	12.07
	2.0	0.93	0.18	0.11	62.02	0.93	0.11	- 0.91	19.50	0.93	0.08	- 1.38	12.07
	2.5	0.98	0.19	0.11	62.02	0.97	0.12	- 0.91	19.50	0.97	0.08	- 1.38	12.07
	3.0	1.02	0.21	0.11	62.02	1.00	0.13	- 0.91	19.50	0.99	0.09	- 1.38	12.07
	3.5	1.05	0.22	0.11	62.02	1.03	0.13	- 0.91	19.50	1.02	0.09	- 1.38	12.07
	4.0	1.09	0.24	0.11	62.02	1.06	0.14	- 0.91	19.50	1.04	0.09	- 1.38	12.07

Table II: Summary measures for distribution of upper records

	γ							
	$\gamma = 4.5$				$\gamma = 5.0$			
n	Mean	Var	Sk	Ku	Mean	Var	Sk	Ku
2	1.65	0.32	10.48	487.45	1.56	0.18	13.54	244.29
3	2.13	1.20	4.24	415.39	1.95	0.72	3.16	124.99
4	2.73	2.95	4.12	684.72	2.44	1.69	2.41	143.44
5	3.51	6.50	4.89	1337.72	3.05	3.51	2.63	194.15
6	4.52	13.58	6.05	2827.78	3.81	6.86	3.13	279.29
7	5.81	27.47	7.50	6282.35	4.77	12.97	3.77	414.95
8	7.47	54.43	9.27	14474.10	5.96	24.00	4.53	630.42
9	9.60	106.18	11.39	34304.80	7.45	43.71	5.39	975.14
10	12.34	204.68	13.94	83169.60	9.31	78.64	6.37	1531.86
11	15.87	390.81	17.00	205396.0	11.64	140.11	7.47	2439.42
12	20.40	740.47	20.70	515028.0	14.55	247.63	8.73	3932.00
13	26.23	1394.03	25.18	1307920	18.19	434.78	10.16	6406.46
14	33.73	2610.39	30.59	3357170	22.74	759.11	11.77	10538.60
15	43.37	4865.88	37.17	8965810	28.42	1319.03	13.61	17484.40
	$\gamma = 5.5$				$\gamma = 6.0$			
n	Mean	Var	Sk	Ku	Mean	Var	Sk	Ku
2	1.49	0.10	20.64	193.99	1.44	0.06	36.43	174.95
3	1.83	0.46	2.81	70.34	1.73	0.32	2.83	48.95
4	2.23	1.07	1.57	69.13	2.07	0.72	1.11	44.33
5	2.73	2.11	1.52	81.00	2.49	1.38	0.87	48.13
6	3.33	3.93	1.76	100.60	2.99	2.46	0.97	55.41
7	4.07	7.05	2.10	128.22	3.58	4.24	1.17	65.37
8	4.98	12.38	2.52	165.92	4.30	7.13	1.43	78.09
9	6.09	21.39	2.98	217.08	5.16	11.82	1.72	94.03
10	7.44	36.48	3.49	286.63	6.19	19.33	2.03	113.91
11	9.09	61.63	4.05	381.55	7.43	31.29	2.37	138.69
12	11.11	103.26	4.66	511.72	8.92	50.25	2.74	169.62
13	13.58	171.85	5.33	691.08	10.70	80.14	3.13	208.33
14	16.60	284.37	6.07	939.39	12.84	127.08	3.54	256.88
15	20.29	468.24	6.87	1284.71	15.41	200.52	3.99	317.96

Var: variance; Sk: skewness; Ku: kurtosis

Table III: Summary measures for distribution of lower records

	γ							
	γ = 4.5				γ = 5.0			
n	Mean	Var	Sk	Ku	Mean	Var	Sk	Ku
2	0.728	0.065	- 2.661	3.136	0.747	0.057	- 2.762	3.127
3	0.572	0.046	- 2.605	3.151	0.601	0.042	- 2.712	3.061
4	0.458	0.035	- 2.547	3.101	0.492	0.033	- 2.659	2.971
5	0.371	0.027	- 2.456	3.147	0.406	0.027	- 2.573	2.979
6	0.302	0.021	- 2.343	3.302	0.337	0.022	- 2.467	3.085
7	0.246	0.016	- 2.220	3.549	0.280	0.017	- 2.352	3.268
8	0.201	0.013	- 2.092	3.873	0.233	0.014	- 2.235	3.516
9	0.164	0.010	- 1.963	4.267	0.194	0.011	- 2.117	3.818
10	0.135	0.007	- 1.834	4.725	0.162	0.009	- 2.001	4.169
11	0.110	0.005	- 1.705	5.246	0.135	0.007	- 1.886	4.566
12	0.090	0.004	- 1.576	5.828	0.112	0.005	- 1.773	5.006
13	0.074	0.003	- 1.448	6.473	0.093	0.004	- 1.661	5.490
14	0.060	0.002	- 1.320	7.183	0.078	0.003	- 1.550	6.017
15	0.049	0.002	- 1.191	7.963	0.065	0.002	- 1.439	6.589
	γ							
	γ = 5.5				γ = 6.0			
n	Mean	Var	Sk	Ku	Mean	Var	Sk	Ku
2	0.765	0.050	- 2.853	3.156	0.780	0.045	- 2.937	3.218
3	0.627	0.038	- 2.804	3.017	0.649	0.035	- 2.884	3.007
4	0.522	0.031	- 2.752	2.895	0.548	0.029	- 2.831	2.854
5	0.438	0.026	- 2.669	2.874	0.466	0.025	- 2.751	2.810
6	0.369	0.022	- 2.569	2.943	0.398	0.022	- 2.655	2.849
7	0.311	0.018	- 2.461	3.080	0.341	0.018	- 2.553	2.952
8	0.263	0.015	- 2.352	3.272	0.292	0.015	- 2.450	3.102
9	0.223	0.012	- 2.243	3.510	0.250	0.013	- 2.348	3.292
10	0.188	0.010	- 2.137	3.787	0.214	0.011	- 2.249	3.515
11	0.159	0.008	- 2.033	4.099	0.184	0.009	- 2.153	3.766
12	0.135	0.006	- 1.930	4.444	0.157	0.007	- 2.060	4.044
13	0.114	0.005	- 1.830	4.821	0.135	0.006	- 1.968	4.346
14	0.096	0.004	- 1.731	5.228	0.116	0.004	- 1.879	4.671
15	0.082	0.003	- 1.634	5.667	0.099	0.004	- 1.791	5.019

Var: variance; Sk: skewness; Ku: kurtosis

Figure I: $S_n(x)$ for $\{n = 1,2\}$, $\{\beta_1 = 1,2,3,4\}$, $\{\beta_2 = 1,2,3\}$ and $\{\gamma = 2,3\}$

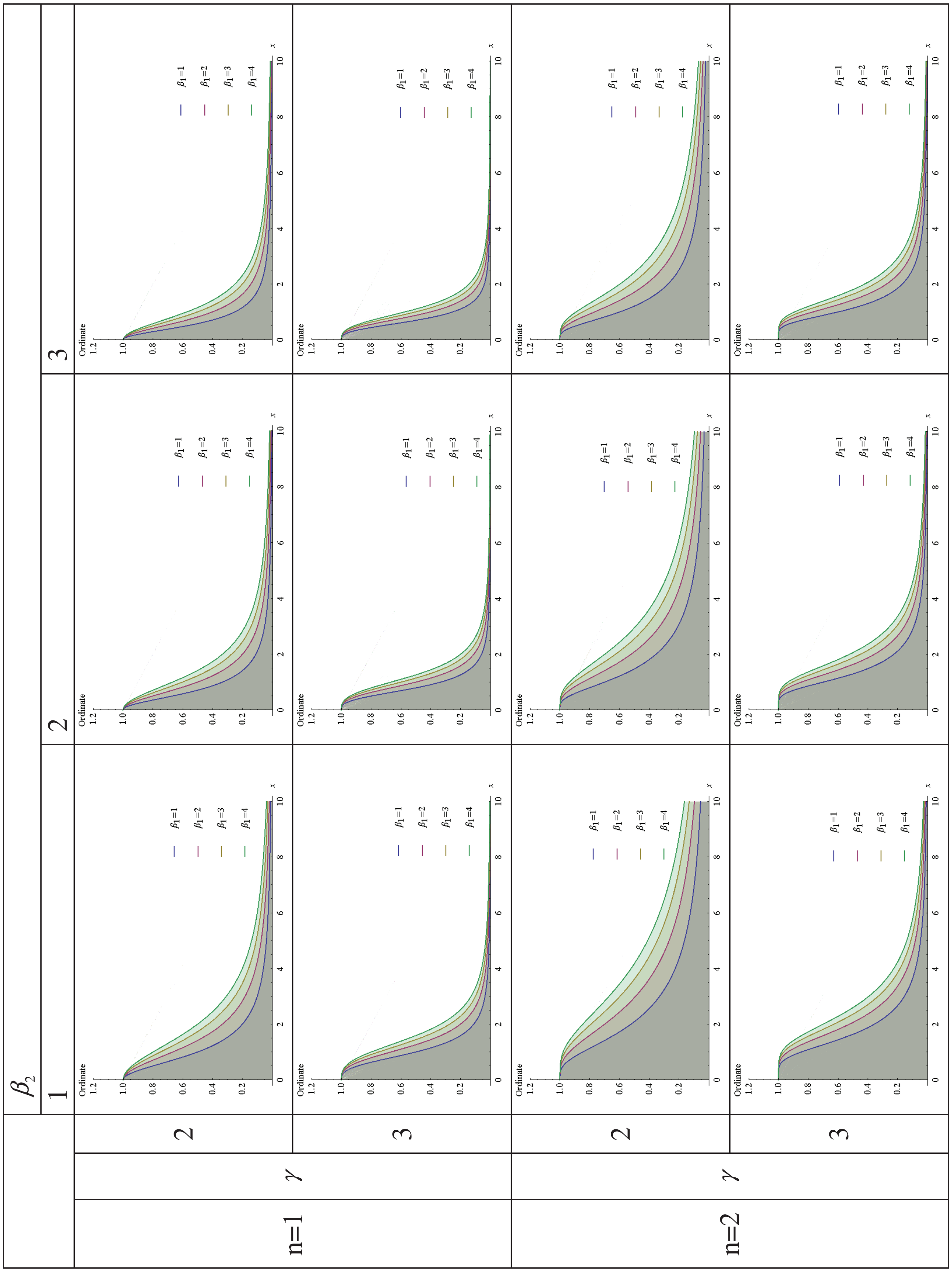

Figure II: $h_n(x)$ for $\{n = 1,2\}$, $\{\beta_1 = 1,2,3,4\}$, $\{\beta_2 = 1,2,3\}$ and $\{\gamma = 2,3\}$