

## RESEARCH ARTICLE

# $C^3$ cubic trigonometric B-spline curves with a real parameter

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**Abstract:** This paper presents a class of cubic trigonometric B-spline curves with a real parameter  $\alpha$ , which are called  $\alpha$ -B-spline curves. The  $\alpha$ -B-spline curves have properties similar to those of the standard cubic uniform B-spline curves. In particular, the  $\alpha$ -B-spline curves are  $C^3$  and can achieve shape adjustment by altering the value of the parameter  $\alpha$  even if the control points are kept unchanged. With proper conditions, the  $\alpha$ -B-spline curves also can exactly represent arcs of ellipses and parabolas. In order to obtain smooth  $\alpha$ -B-spline curves, a method for determining the value of parameter  $\alpha$  is presented.

**Keywords:** B-spline curve,  $C^3$  continuity, shape adjustment, trigonometric polynomial.

## INTRODUCTION

In geometric modelling, it is often necessary to freely change the shapes of curves. Hence, the curves with parameters have been paid more and more attention by many researchers in recent years. For examples: the polynomial Bézier curves with parameters (Yang & Zeng, 2009; Yan & Liang, 2011a; Chen & Wang, 2011; Qin *et al.*, 2013); trigonometric Bézier curves with parameters (Han *et al.*, 2009; Bashir *et al.*, 2013; Bashir & Ali, 2016; Li, 2013; Misro *et al.*, 2017); polynomial B-spline curves with parameters (Cao & Wang, 2011; Han, 2011; Zhu *et al.*, 2015); trigonometric B-spline curves with parameters (Juhász & Roth, 2010; Yan & Liang, 2011b; Han & Zhu, 2012; Dube & Sharma, 2014) and hyperbolic B-spline curves with parameters (Liu *et al.*, 2010). Those curves with parameters inherit the similar to or the same properties as the corresponding standard curves, but also

have better performance ability because of the parameters. The shapes of these curves can be adjusted by altering the values of the parameters even if the control points remain unchanged, which is difficult to achieve for the corresponding standard curves. Moreover, some of these curves can exactly represent ellipses and parabolas without complicated conditions. Therefore, the curves with parameters have wide applications in engineering.

It is known that the B-spline technique is one of the methods of analytic representation of curves and surfaces that have won wide acceptance as a valuable tool in CAD/CAM systems. They are used to produce curves, which appear reasonably smooth at all scales. At present, many CAD/CAM systems employ B-spline curves as their major building blocks, since they can attain a number of mathematical properties. The main purpose of this paper is to present a class of newly constructed cubic trigonometric B-spline curves with a parameter. The proposed curves not only inherit all geometric properties of the standard cubic uniform B-spline curves, but also are  $C^3$  and can be adjusted by altering the value of the parameter even if the control points are fixed. With proper conditions, the  $\alpha$ -B-spline curves can also exactly represent ellipses and parabolas.

## METHODOLOGY

First, the cubic trigonometric B-spline basis functions with a parameter  $\alpha$  ( $\alpha$ -B-spline basis functions for short) are defined as follows;

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**Definition 1.** For  $0 \leq t \leq 1$ ,  $0.3 \leq \alpha \leq 0.5$ , the following functions about  $t$  are called the  $\alpha$ -B-spline basis functions,

$$\begin{cases} f_0(t) = \frac{1}{2}((4\alpha - 1) - 2\alpha S + (3 - 8\alpha)S^2 + (6\alpha - 2)S^3) \\ f_1(t) = \frac{1}{2}((2 - 4\alpha) + 2\alpha C + (8\alpha - 3)S^2 + (2 - 6\alpha)C^3) \\ f_2(t) = \frac{1}{2}((4\alpha - 1) + 2\alpha S + (3 - 8\alpha)S^2 + (2 - 6\alpha)S^3) \\ f_3(t) = \frac{1}{2}((2 - 4\alpha) - 2\alpha C + (8\alpha - 3)S^2 + (6\alpha - 2)C^3) \end{cases} \dots(1)$$

where  $S := \sin \frac{\pi}{2}t$ ,  $C := \cos \frac{\pi}{2}t$ .

By simple deduction, the  $\alpha$ -B-spline basis functions have the following properties:

- (a) Nonnegativity:  $f_i(t) \geq 0$  ( $i = 0, 1, 2, 3$ ).
- (b) Mixing:  $\sum_{i=0}^3 f_i(t) \equiv 1$ .
- (c) Symmetricity:  $f_i(1-t) = f_{3-i}(t)$  ( $i = 0, 1, 2, 3$ ).
- (d) Endpoint properties: the  $\alpha$ -B-spline basis functions have the following properties at the endpoints:

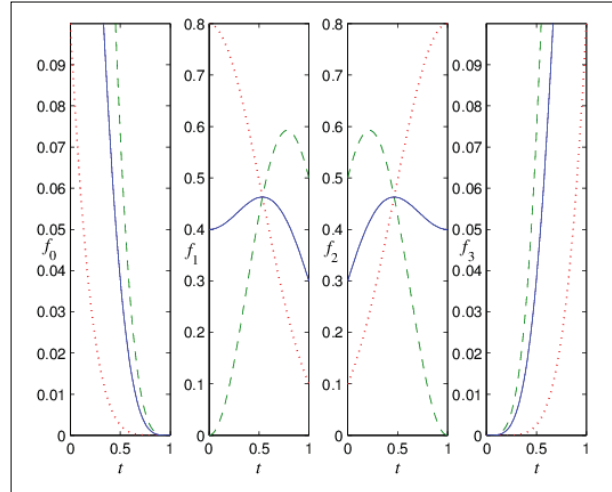
$$\begin{cases} f_0(0) = \frac{1}{2}(4\alpha - 1), f_1(0) = -4\alpha + 2, f_2(0) = \frac{1}{2}(4\alpha - 1), f_3(0) = 0 \\ f_0(1) = 0, f_1(1) = \frac{1}{2}(4\alpha - 1), f_2(1) = -4\alpha + 2, f_3(1) = \frac{1}{2}(4\alpha - 1) \end{cases}$$

$$\begin{cases} f_0'(0) = -\frac{\pi}{2}\alpha, f_1'(0) = 0, f_2'(0) = \frac{\pi}{2}\alpha, f_3'(0) = 0 \\ f_0'(1) = 0, f_1'(1) = -\frac{\pi}{2}\alpha, f_2'(1) = 0, f_3'(1) = \frac{\pi}{2}\alpha \end{cases}$$

$$\begin{cases} f_0''(0) = \frac{\pi^2}{4}(-8\alpha + 3), f_1''(0) = -\frac{\pi^2}{2}(-8\alpha + 3), f_2''(0) = \frac{\pi^2}{4}(-8\alpha + 3), f_3''(0) = 0 \\ f_0''(1) = 0, f_1''(1) = \frac{\pi^2}{4}(-8\alpha + 3), f_2''(1) = -\frac{\pi^2}{2}(-8\alpha + 3), f_3''(1) = \frac{\pi^2}{4}(-8\alpha + 3) \end{cases}$$

$$\begin{cases} f_0'''(0) = \frac{\pi^3}{8}(19\alpha - 6), f_1'''(0) = 0, f_2'''(0) = -\frac{\pi^3}{8}(19\alpha - 6), f_3'''(0) = 0 \\ f_0'''(1) = 0, f_1'''(1) = \frac{\pi^3}{8}(19\alpha - 6), f_2'''(1) = 0, f_3'''(1) = -\frac{\pi^3}{8}(19\alpha - 6) \end{cases}$$

- (e) Shape-adjustability: a family of curves of  $\alpha$ -B-spline basis functions controlled by parameter  $\alpha$  can be obtained. Figure 1 shows curves of the  $\alpha$ -B-spline basis functions with  $a = 0.3$  (dotted lines),  $a = 0.4$  (solid lines) and  $a = 0.5$  (dashed lines).



**Figure 1:** Curves of the  $\alpha$ -B-spline basis functions for different  $a$

Note that the value of the parameter  $a$  can actually be taken as any real number. But the  $\alpha$ -B-spline basis functions would not be nonnegative when  $\alpha < 0.3$  or  $\alpha > 0.3$ , which can be illustrated by drawing the curves of the basis functions. Thus, the parameter is taken as  $\alpha \in [0.3, 0.5]$  to ensure that the  $\alpha$ -B-spline basis functions are nonnegative.

Then, the following cubic trigonometric B-spline curves with a parameter  $\alpha$  ( $\alpha$ -B-spline curves for short) can be defined based on the  $\alpha$ -B-spline basis functions.

**Definition 2.** Given  $n+1$  control points  $\mathbf{p}_k$  ( $k = 0, 1, \dots, n$ ) in  $R^2$  or  $R^3$ , for  $0 \leq t \leq 1$ ,  $0.3 \leq \alpha \leq 0.5$ , the following piecewise curves are called  $\alpha$ -B-spline curves,

$$\mathbf{r}_i(t) = \sum_{j=0}^3 f_j(t) \mathbf{p}_{i+j}, \dots(2)$$

where  $i = 0, 1, \dots, n-3$ ,  $f_j(t)$  ( $j = 0, 1, 2, 3$ ) are the  $\alpha$ -B-spline basis functions.

## RESULTS AND DISCUSSION

From properties of the  $\alpha$ -B-spline basis functions, some properties of  $\alpha$ -B-spline curves can be obtained as follows:

- (a) Geometric invariance: because equation (2) is a vector function, shape of  $\alpha$ -B-spline curves is determined by the control points and parameter  $\alpha$  and has nothing to do with the coordinate system.

(b) Convex hull and convex-preserving: from the nonnegativity and mixing of the  $\alpha$ -B-spline basis functions, a piece of  $\alpha$ -B-spline curves  $r_i(t)$  must lie inside its control polygon spanned by  $p_{i+j}$  ( $j=0,1,2,3$ ). Furthermore, a piece of  $\alpha$ -B-spline curves  $r_i(t)$  is convex if its control polygon is convex.

(c) Symmetricity: for fixed  $\alpha$ , from the symmetricity of the  $\alpha$ -B-spline basis functions and equation (2),

$$r_i(1-t; p_{i+3}, p_{i+2}, p_{i+1}, p_i) = r_i(t; p_i, p_{i+1}, p_{i+2}, p_{i+3}) \quad \dots(3)$$

Equation (3) shows that both  $p_{i+j}$  ( $j=0,1,2,3$ ) and  $p_{i+3-j}$  ( $j=0,1,2,3$ ) define the same piece of  $\alpha$ -B-spline curves in a different parameterisation.

(d)  $C^3$  continuity: from the endpoint properties of the  $\alpha$ -B-spline basis functions and equation (2),

$$\begin{cases} r_i(0) = \frac{1}{2}((4\alpha-1)p_i + (-8\alpha+4)p_{i+1} + (4\alpha-1)p_{i+2}) \\ r_i(1) = \frac{1}{2}((4\alpha-1)p_{i+1} + (-8\alpha+4)p_{i+2} + (4\alpha-1)p_{i+3}) \end{cases} \quad \dots(4)$$

$$\begin{cases} r_i'(0) = \frac{\pi}{2}\alpha(p_{i+2} - p_i) \\ r_i'(1) = \frac{\pi}{2}\alpha(p_{i+3} - p_{i+1}) \end{cases} \quad \dots(5)$$

$$\begin{cases} r_i''(0) = \frac{\pi^2}{4}(-8\alpha+3)(p_i - 2p_{i+1} + p_{i+2}) \\ r_i''(1) = \frac{\pi^2}{4}(-8\alpha+3)(p_{i+1} - 2p_{i+2} + p_{i+3}) \end{cases} \quad \dots(6)$$

$$\begin{cases} r_i'''(0) = \frac{\pi^3}{8}(19\alpha-6)(p_i - p_{i+2}) \\ r_i'''(1) = \frac{\pi^3}{8}(19\alpha-6)(p_{i+1} - p_{i+3}) \end{cases} \quad \dots(7)$$

By equation (4) ~ equation (7), then

$$r_i^{(k)}(1) = r_{i+1}^{(k)}(0) \quad (k=0,1,2,3) \quad \dots(8)$$

Equation (8) shows that the  $\alpha$ -B-spline curves are  $C^3$ .

(e) Shape-adjustability: for fixed control points  $p_{i+j}$  ( $j=0,1,2,3$ ), suppose  $p_i$  and  $p_{i+3}$  lie on the same side of edge  $p_{i+1}p_{i+2}$ . Let  $p^* = \frac{p_{i+1} + p_{i+2}}{2}$ , from equation (2),

$$\left\| r_i\left(\frac{1}{2}\right) - p^* \right\| = \frac{\sqrt{2\alpha+1}-\sqrt{2}}{4} \|p_i - p_{i+1} - p_{i+2} + p_{i+3}\| \quad \dots(9)$$

where  $\|\cdot\|$  represents the norm of a vector.

When control points  $p_{i+j}$  ( $j=0,1,2,3$ ) are fixed,  $\|p_i - p_{i+1} - p_{i+2} + p_{i+3}\|$  in equation (9) would keep unchanged. Because  $\frac{\sqrt{2\alpha+1}-\sqrt{2}}{4}$  increases as  $\alpha$  increases, a piece of  $\alpha$ -B-spline curves defined by  $p_{i+j}$  ( $j=0,1,2,3$ ) would be more approximate to its control polygon with the decrease of  $\alpha$ . With proper value of the parameter  $\alpha$ , a piece of  $\alpha$ -B-spline curves defined by  $p_{i+j}$  ( $j=0,1,2,3$ ) can be more approximate to its control polygon than a piece of standard cubic uniform B-spline curves defined by the same control points (Figure 2).

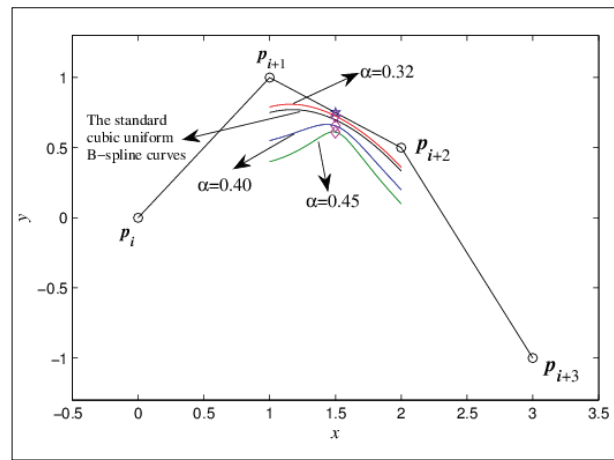


Figure 2: Different shapes of a piece of  $\alpha$ -B-spline curves

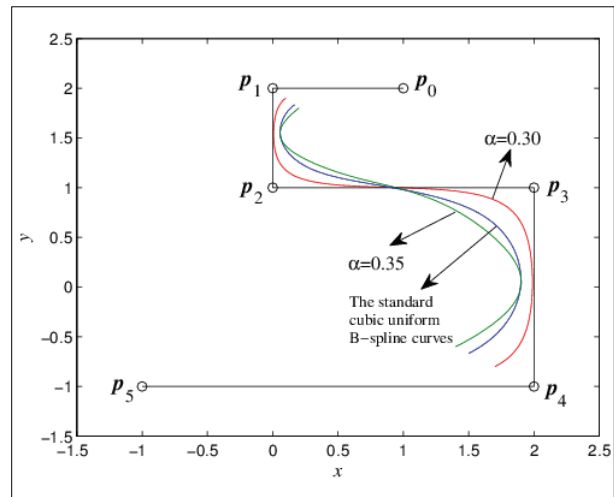


Figure 3: Different shapes of  $C^3$   $\alpha$ -B-spline curves

Furthermore, for fixed control points  $p_i (i = 0, 1, \dots, n)$ , shape of the  $C^3$   $\alpha$ -B-spline curves can be adjusted by altering the value of parameter  $\alpha$ . With proper value of the parameter  $\alpha$ , the  $C^3$   $\alpha$ -B-spline curves can be more approximate to the control polygon than the standard cubic uniform B-spline curves (Figure 3).

Moreover, the  $\alpha$ -B-spline curves can exactly represent arcs of ellipse and parabola as follows:

(a) Let  $p_i = \left(-\frac{3a}{2}, 0\right)$ ,  $p_{i+1} = \left(0, \frac{3b}{2}\right)$ ,  $p_{i+2} = \left(\frac{3a}{2}, 0\right)$ ,  $p_{i+3} = \left(0, -\frac{3b}{2}\right)$ ,  $ab \neq 0$ ,  $\alpha = \frac{1}{3}$ . Then, equation (2) can be rewritten as,

$$r_i(t) = \left(a \sin \frac{\pi t}{2}, b \cos \frac{\pi t}{2}\right), 0 \leq t \leq 1. \quad \dots(10)$$

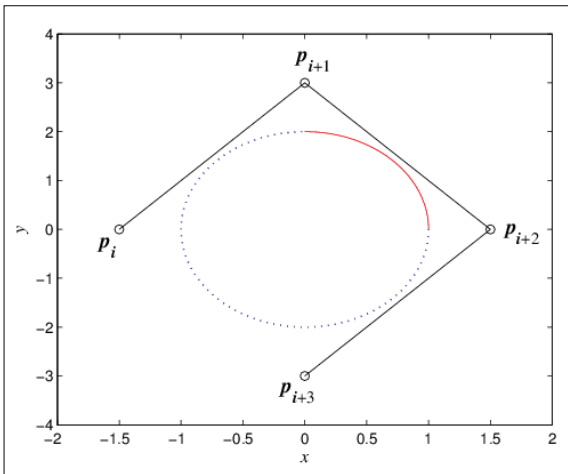


Figure 4: The exact representation of arcs of ellipse

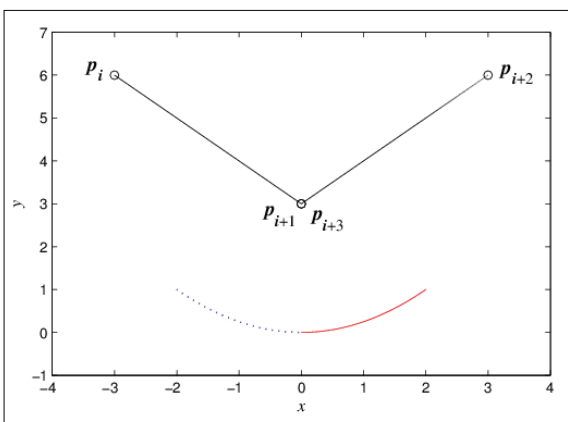


Figure 5: The exact representation of arcs of parabola

When  $a = b$ , equation (10) is the expression of arcs of circle; when  $a \neq b$ , equation (10) shows arcs of ellipse. Figure 4 shows the exact representation of arcs of ellipse by the  $\alpha$ -B-spline curves for  $a = 1$  and  $b = 2$ .

(b) Let  $p_i = \left(-\frac{3a}{2}, 6b\right)$ ,  $p_{i+1} = (0, 3b)$ ,  $p_{i+2} = \left(\frac{3a}{2}, 6b\right)$ ,  $p_{i+3} = (0, 3b)$ ,  $ab \neq 0$ ,  $\alpha = \frac{1}{3}$ . Then, equation (2) can be rewritten as,

$$r_i(t) = \left(a \sin \frac{\pi t}{2}, b \sin^2 \frac{\pi t}{2}\right), 0 \leq t \leq 1. \quad \dots(11)$$

Equation (11) shows arcs of parabola. Figure 5 shows the exact representation of arcs of parabola by the  $\alpha$ -B-spline curves for  $a = 2$  and  $b = 1$ .

**Remark 1.** It is clear that the  $\alpha$ -B-spline curves have properties similar to those of the standard cubic uniform B-spline curves. In contrast with the standard cubic uniform B-spline curves, the  $\alpha$ -B-spline curves presented in this work have the following characteristics,

- (a) When the control points are fixed, the standard cubic uniform B-spline curves are only  $C^2$  and unique, while the  $\alpha$ -B-spline curves are  $C^3$  and can be adjusted by altering value of the parameter  $\alpha$ .
- (b) The standard cubic uniform B-spline curves cannot exactly represent arcs of ellipse and arcs of parabola, while the  $\alpha$ -B-spline curves have this ability.
- (c) With proper value of the parameter  $\alpha$ , the  $\alpha$ -B-spline curves can be a better approximation to the control polygon than the standard cubic uniform B-spline curves.

As mentioned above, shape of the  $\alpha$ -B-spline curves can be adjusted by the parameter  $\alpha$  even if the control points are kept unchanged. Hence, smoothness of the  $\alpha$ -B-spline curves is mainly determined by the parameter  $\alpha$  when the control points are fixed. If the parameter  $\alpha$  is chosen improperly, the corresponding  $\alpha$ -B-spline curves would be unsatisfied. Figure 6 shows the  $\alpha$ -B-spline curves with  $\alpha = 0.4$  for the same control points in Figure 3.

It is clear that the  $\alpha$ -B-spline curves in Figure 3 are smoother than those in Figure 6. Therefore, smoothing problem of the  $\alpha$ -B-spline curves needs to be considered when choosing value of the parameter  $\alpha$ .

According to smoothing criterion (Poliakoff, 1996), for fixed control points  $p_k (k = 0, 1, \dots, n)$ , smoothness of the  $\alpha$ -B-spline curves  $r_i(t) (i = 0, 1, \dots, n - 3)$  is mainly

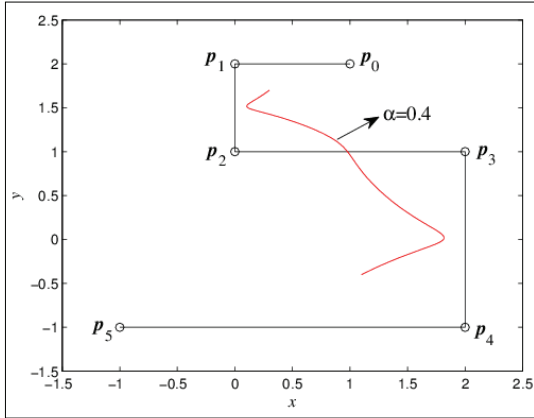


Figure 6: The  $\alpha$ -B-spline curves with improper  $\alpha$

determined by the energy value of the curves, which can be expressed as,

$$E = \sum_{i=0}^{n-3} \int_0^1 \|r_i''(t)\|^2 dt \quad \dots(12)$$

To make the  $\alpha$ -B-spline curves as smooth as possible, one may need to determine the value of the parameter  $\alpha$  to minimise the energy value of the curves. Hence, an optimisation model for determining the value of the parameter  $\alpha$  can be obtained as,

$$\begin{aligned} \min F(\alpha) &= \sum_{i=0}^{n-3} \int_0^1 \|r_i''(t)\|^2 dt \\ \text{s.t. } &0.3 \leq \alpha \leq 0.5 \end{aligned} \quad \dots(13)$$

Let

$$\begin{aligned} M_0(t) &= \frac{1}{2}(4 - 2S - 8S^2 + 6S^3), & N_0(t) &= \frac{1}{2}(-1 + 3S^2 - 2S^3), \\ M_1(t) &= \frac{1}{2}(-4 + 2C + 8S^2 - 6C^3), & N_1(t) &= \frac{1}{2}(2 - 3S^2 + 2C^3), \\ M_2(t) &= \frac{1}{2}(4 + 2S - 8S^2 - 6S^3), & N_2(t) &= \frac{1}{2}(-1 + 3S^2 + 2S^3), \\ M_3(t) &= \frac{1}{2}(-4 - 2C + 8S^2 + 6C^3), & N_3(t) &= \frac{1}{2}(2 - 3S^2 - 2C^3), \end{aligned}$$

where  $S := \sin \frac{\pi}{2}t$ ,  $C := \cos \frac{\pi}{2}t$ ,  $0 \leq t \leq 1$ .

Then, equation (2) can be rewritten as,

$$r_i(t) = H_i(t)\alpha + G_i(t) \quad \dots(14)$$

where  $H_i(t) = \sum_{j=0}^3 M_j(t)p_{i+j}$ ,  $G_i(t) = \sum_{j=0}^3 N_j(t)p_{i+j}$ .

From equation (14), equation (12) can be rewritten as,

$$E = A\alpha^2 + 2B\alpha + C \quad \dots(15)$$

$$\begin{aligned} \text{where } A &= \sum_{i=0}^{n-3} \int_0^1 \|H_i''(t)\|^2 dt, & B &= \sum_{i=0}^{n-3} \int_0^1 (H_i''(t) \cdot G_i''(t)) dt, \\ C &= \sum_{i=0}^{n-3} \int_0^1 \|G_i''(t)\|^2 dt. \end{aligned}$$

It can be easily calculated that the extreme point of equation (15) is  $\alpha = -\frac{B}{A}$  ( $A \neq 0$ ). Then, the solution of equation (13) has two cases:

(i) when  $-\frac{B}{A} < 0.3$  or  $-\frac{B}{A} > 0.5$ : if  $F(0.3) \leq F(0.5)$ , the solution of equation (13) is  $\alpha = 0.3$ ; else, the solution of equation (13) is  $\alpha = 0.5$ .

(ii) when  $0.3 \leq -\frac{B}{A} \leq 0.5$ : if  $F(0.3) = \min\{F(0.3), F(0.5), F(-\frac{B}{A})\}$ , the solution of equation (13) is  $\alpha = 0.3$ ; else if  $F(0.5) = \min\{F(0.3), F(0.5), F(-\frac{B}{A})\}$ , the solution of equation (13) is  $\alpha = 0.5$ ; else, the solution of equation (13) is  $\alpha = -\frac{B}{A}$ .

Figure 7 shows the smoothest  $\alpha$ -B-spline curves for the same control points in Figure 3, where the value of the parameter  $\alpha$  found using equation (13) is  $\alpha = 0.3375$ .

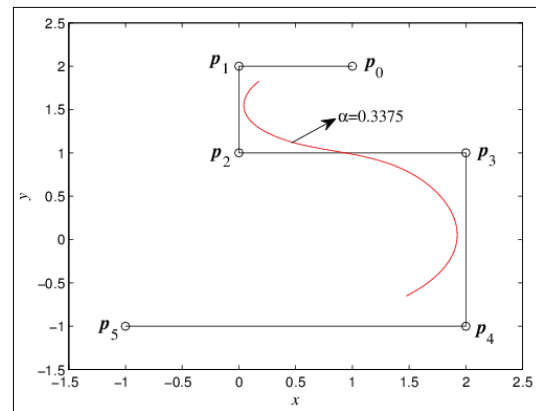


Figure 7: The smoothest  $\alpha$ -B-spline curves

## CONCLUSION

The  $\alpha$ -B-spline curves presented in this paper not only have properties similar to the standard cubic uniform B-spline curves, but also are  $C^3$  and can be adjusted by altering the value of the parameter even if the control points are fixed. With proper conditions, the  $\alpha$ -B-spline

curves can also exactly represent arcs of ellipse and parabola. In order to make sure the  $\alpha$ -B-spline curves are as smooth as possible, a method for determining the value of the parameter is presented. Because there are no differences in structure between the  $\alpha$ -B-spline curves and the standard cubic uniform B-spline curves, it is not difficult to adapt the  $\alpha$ -B-spline curves to the CAD/CAM systems that already use the corresponding standard cubic uniform B-spline curves.

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