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RESEARCH ARTICLE

The size, multipartite Ramsey numbers for C_3 versus all graphs up to 4 vertices

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Abstract: In this paper we restrict our attention to finite graphs containing no loops or multiple edges. The multipartite graph $K_{j\times s}$ ($j \geq 3$) consisting of j partite sets of uniform size s is defined as $V(K_{j\times s}) = \{v_{mn} \mid m \in \{1,2,...,j\}$ and $n \in \{1,2,...,s\}\}$ and $E(K_{j\times s}) = \{v_{mn} v_{kl} \mid m,k \in \{1,2,...,j\}$ and $n, l \in \{1,2,...,s\}$ where $k \neq m\}$. The set of vertices of the m^{th} partite set is denoted by $\{v_{mn} \mid n \in \{1,2,...,s\}\}$. If for every two-colouring (red and blue) of the edges of a graph K, there exists a copy of H in the first colour (red) or a copy of G in the second colour (blue), we write $K \rightarrow (H,G)$. Given two simple graphs H and G, the Ramsey number $r(H,G)$ is defined as the smallest positive integer s such that $K_s \rightarrow (H,G)$ and along the same line of reasoning, the multipartite Ramsey number $m_j(H,G)$ is defined as the smallest positive integer s such that $K_{j\times s} \rightarrow (H,G)$. Thus, multipartite Ramsey number $m_j(C_3,G)$ is defined as the smallest positive integer s such that any red-blue colouring of $K_{j\times s}$ contains a red C_3 or a blue G. Since only a few multipartite Ramsey numbers for pairs of graphs have been found so far, in this paper we find all such multipartite Ramsey numbers $m_j(C_3,G)$ when G is any graph up to 4 vertices.

Keywords: Combinatorics, graph theory, mathematics, multipartite Ramsey numbers, Ramsey theory.

INTRODUCTION

Assume that the multipartite graph $K_{j\times s}$ (also denoted by $K_{s,s,...s}$) consists of j partite sets of uniform size s. The concept of multipartite Ramsey numbers related to two-colourings of $K_{j\times s}$ was introduced by Burger and van Vuuren (2004). Consider any red/blue colouring of $K_{j\times s}$. Let H_R denote the graph having the vertex set $V(K_{j\times s})$ and the edge set consisting of all the red edges. Similarly let H_B denote the graph having the vertex set $V(K_{j\times s})$ and the edge set consisting of all the blue edges. We denote such an edge colouring by $K_{j\times s} = H_R \oplus H_B$. For the pair of graphs H and G, the multipartite Ramsey number $m_j(H,G)$ is defined as the smallest positive integer s such that any red-blue colouring of $K_{j\times s}$ where $K_{j\times s} = H_R \oplus H_B$ contains a red H (i.e. H is a subgraph of H_R) or a blue G. However, it is worth noting that if such a number doesn't exist we denote it by ∞ as in the bipartite case of $m_2(C_3,C_3)$. Currently, only a few multipartite Ramsey numbers for pairs of graphs have been found (Faudree & Schelp, 1975; Syafrizal *et al*., 2007). In this paper, we find all such multipartite Ramsey numbers $m_j(C_3,G)$ where G is any isolate-free graph up to 4 vertices as illustrated in Table 1. Note that $C_3 = K_3$ and $B_2 = K_4 - e$.

The entries of columns corresponding to $G = P_2$, $G = 2K_2$ and $G = P_3$ of Table 1 follow from Syafrizal *et al.* (2007) and Chvátal and Harary (1972). Theorem 1 deals with the entries of the column corresponding to $G = 2K_2$. Theorems 2, 3, 4, 5, 6 and 7 deal with the entries of the columns corresponding to $G = K_{1,3}$, $G = C_3$, $G = C_4$, $G = K_{1,3} + x$, $G = B_2$ and $G = K_4$, respectively.

Table 1: Values of $m_j(C_3,G)$

	$m_j(C_3,G)$ values									
$G =$	P_2	$2K_2$	P_3	P_4	$K_{1,3}$	C_3	C_4	$K_{1,3}+x$	B_2	K_4
$j = 3$	1	2	2	3	3	∞	3	∞	∞	∞
$j = 4$	1	2	2	2	3	∞	2	∞	∞	∞
$j = 5$	1	1	1	2	2	∞	2	∞	∞	∞
$j = 6$	1	1	1	1	2	1	2	2	2	∞
$j = 7$	1	1	1	1	1	1	1	1	1	∞
$j = 8$	1	1	1	1	1	1	1	1	1	∞
$j \geq 9$	1	1	1	1	1	1	1	1	1	1

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The final section deals with how to extend Table 1 to obtain multipartite Ramsey numbers for C_3 *versus* any graph up to 4 vertices.

$m_j(C_3, G)$ WHEN ENTRIES OF COLUMN CORRESPONDING TO $G = 2K_2$

Theorem 1. *If* $j \geq 3$, *then*

$$m_j(C_3, 2K_2) = \begin{cases} 2 & if\ j \in \{3,4\}, \\ 1 & if\ j \geq 5. \end{cases}$$

Proof. Consider the colouring of $K_{4\times 1} = H_R \oplus H_B$ where $v_{11}v_{21}$, $v_{11}v_{31}$, $v_{21}v_{31}$ are blue and all the other edges are red. Then $K_{4\times 1}$ has neither a red C_3 nor a blue $2K_2$. So $m_4(C_3,2K_2) \geq 2$.

Next, consider any colouring of $K_{3\times 2} = H_R \oplus H_B$. If $K_{3\times 2}$ has a red C_3, then we are done. So, assume $K_{3\times 2}$ has no red C_3. Then each of the subgraphs of H_B induced by $\{v_{11},v_{12},v_{13}\}$ and $\{v_{12},v_{22},v_{32}\}$, has blue P_2. Hence, $K_{3\times 2}$ has a blue $2K_2$. Therefore, $m_3(C_3,2K_2) \leq 2$. But, as $m_3(C_3,2K_2) \geq m_4(C_3,2K_2)$ we can conclude that $m_3(C_3,2K_2) = m_4(C_3,2K_2) = 2$.

Clearly, $m_j(C_3,2K_2) = 1$ when $j \geq 5$ as $r(C_3,2K_2) = 5$ (Chvátal & Harary, 1972).

$m_j(C_3,G)$ FOR ALL CONNECTED GRAPHS G UP TO 4 VERTICES

The theorems 2, 3, 4, 5, 6 and 7 in this section can be stated when $G = K_{1,3}$, $G = C_3$, $G = C_4$, $G = K_{1,3} + x$, $G = B_2$ and $G = K_4$, respectively.

Theorem 2. *If* $j \geq 3$, *then*

$$m_j(C_3, K_{1,3}) = \begin{cases} 3 & if\ j \in \{3,4\}, \\ 2 & if\ j \in \{5,6\}, \\ 1 & if\ j \geq 7. \end{cases}$$

Proof. Colour the graph $K_{4\times 2}$ such that it consists of two disjoint blue cycles of size four, where each of the 4 cycles lies in two partite sets. Colour the remaining edges by red (Figure 1). Then the graph has no red C_3 and has no blue $K_{1,3}$. Therefore, $m_4(C_3,K_{1,3}) \geq 3$.

Now, consider any red C_3 free and blue $K_{1,3}$ free colouring of $K_{3\times 3} = H_R \oplus H_B$. Then H_B has a blue P_4 as $m_3(C_3,P_4) = 3$. Therefore, we have the following two cases.

Case 1: Two vertices of P_4 belong to the same partite set, while the other two vertices belong to two different partite sets.

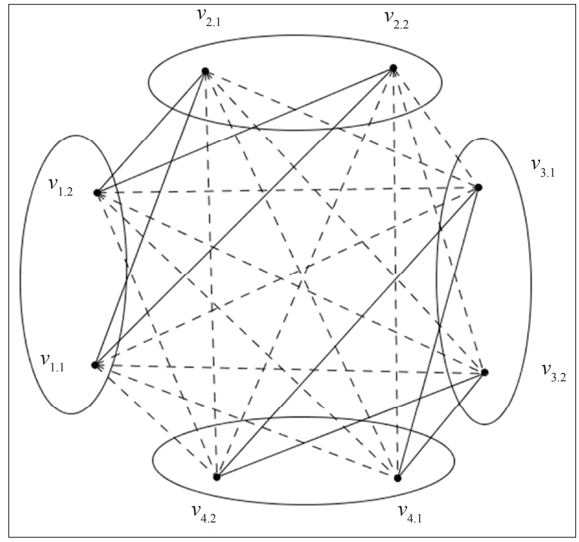

Figure 1: Graph used in proof of theorem 2

Case 2: Two vertices of P_4 belong to the same partite set, while the other two vertices belong to another partite set.

Suppose the graph is as in Case 1. Without loss of generality let the blue P_4 be $v_{11}v_{21}v_{31}v_{12}$. If one of the edges $v_{21}v_{13}, v_{21}v_{33}, v_{21}v_{32}, v_{31}v_{13}, v_{31}v_{22}$ is blue, then the graph has a blue $K_{1,3}$. Therefore, we may assume that all these edges are red. But, then the edges $v_{13}v_{22}, v_{13}v_{32}, v_{13}v_{33}$ will be forced to be blue. Thus, we will get a blue $K_{1,3}$, a contradiction.

Now, suppose that the graph is as in Case 2. Without loss of generality let the blue P_4 be $v_{11}v_{21}v_{12}v_{22}$. If one of the edges $v_{21}v_{31}$, $v_{21}v_{32}$, $v_{21}v_{33}$, $v_{21}v_{13}$ is blue then the graph has a blue $K_{1,3}$. Therefore, we may assume that all these edges are red, and consequently the edges $v_{13}v_{31}$, $v_{13}v_{32}$, $v_{13}v_{33}$ will be forced to be blue. Thus, we will get a blue $K_{1,3}$, a contradiction. Hence, $m_3(C_3,K_{1,3}) \leq 3$. Since $m_3(C_3,K_{1,3}) \geq m_4(C_3,K_{1,3})$, we can conclude that $m_3(C_3,K_{1,3}) = m_4(C_3,K_{1,3}) = 3$.

Consider the colouring of $K_{6\times 1} = H_R \oplus H_B$ where $H_B = 2C_3$. Then the graph has no red C_3 and has no blue $K_{1,3}$. Thus, $m_6(C_3,K_{1,3}) \geq 2$.

Now consider any colouring of $K_{5\times 2} = H_R \oplus H_B$ such that the graph has no red C_3 and has no blue $K_{1,3}$. Then, clearly, $\delta_R(H_R) \geq 6 > \frac{|V(H_R)|}{2}$. Therefore, by Bondy's lemma, which states that any graph G satisfying $\delta(G) > \frac{|V(G)|}{2}$ is pancyclic, we get a red C_3, a contradiction. Hence $m_5(C_3,K_{1,3}) \leq 2$. Next, since $m_5(C_3,K_{1,3}) \geq m_6(C_3,K_{1,3})$, we can conclude that $m_5(C_3,K_{1,3}) = m_6(C_3,K_{1,3}) = 2$.

Clearly, $m_j(C_3,K_{1,3}) = 1$ when $j \geq 7$ as $r(C_3,K_{1,3}) = 7$ (Chvátal & Harary, 1972).

Theorem 3. *If* $j \geq 3$*, then*

$$m_j(C_3, C_3) = \begin{cases} \infty & if\ j \in \{3,4,5\}, \\ 1 & if\ j \geq 6. \end{cases}$$

Proof. For any positive integer s, consider $K_{5\times s}$ where all edges are blue except the edges in the set $A \cup B$ corresponding to,

$$A = \{v_{1p}v_{5q} \mid p, q\ \{1,2,\dots,s\}\} \text{ and}$$

$$B = \bigcup_{m=1}^{4}\{v_{mp}v_{m+1q} \mid p, q\ \{1,2,\dots,s\}\}.$$

Then $K_{5\times s}$ is both red C_3-free and blue C_3-free. Therefore, $m_5(C_3,C_3) = \infty$. Since $m_3(C_3,C_3) \geq m_4(C_3,C_3) \geq m_5(C_3,C_3)$ we have $m_j(C_3,C_3) = \infty$ for $j = 3,4,5$. Also, $m_j(C_3,C_3) = 1$ for $j \geq 6$ since $r(C_3,C_3) = 6$ (Chvátal & Harary, 1972).

Theorem 4. *If* $j \geq 3$*, then*

$$m_j(C_3, C_4) = \begin{cases} 3 & if\ j = 3, \\ 2 & if\ j \in \{4,5,6\}, \\ 1 & if\ j \geq 7. \end{cases}$$

Proof. If $j \geq 7$, since $r(C_3,C_4) = 7$ we get $m_j(C_3,C_4) = 1$. So, we are left with two cases, $j = 3$ and $j \in \{4,5,6\}$.

Case 1: $j = 3$.
Consider the colouring of $K_{3\times 2} = H_R \oplus H_B$ generated by $H_B = 2K_3$. Then, $K_{3\times 2}$ has neither a red C_3 nor a blue C_4. This implies that $m_3(C_3,C_4) \geq 3$. Consider any red/blue colouring given by $K_{3\times 3} = H_R \oplus H_B$, such that H_R contains no red C_3 and H_B contains no blue C_4. Suppose that there exists at least one vertex $v \in K_{3\times 3}$, such that it is incident to 5 or more red edges. Then these 5 vertices belong to two different partite sets. As H_R contains no red C_3 all the edges of the induced subgraph generated by the set $N_R(v)$ will be forced to be blue. Therefore, we will obtain a blue C_4, a contradiction. Hence $\delta(H_B) \geq 2$. This gives rise to two possible scenarios: (i) when H_B has no vertex of blue degree 2, and (ii) when H_B has a vertex of blue degree 2. In the first scenario by handshaking lemma, we note that H_B must have a vertex of blue degree at least 4 and the other vertices must have a blue degree greater than or equal to 3. Let the vertices of the first partite set be denoted by $v_{1,1}, v_{1,2}$ and $v_{1,3}$. Without loss of generality, assume $v_{1,1}$ is a vertex of blue degree 4. Since the blue degrees of $v_{1,2}$ and $v_{1,3}$ are greater than or equal to 3, we get a blue C_4 incident to two vertices of $v_{1,1}, v_{1,2}$ and $v_{1,3}$, a contradiction.

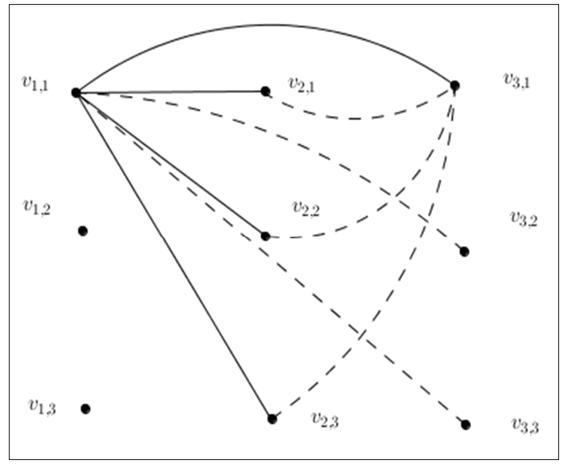

Figure 2: Graph used in the proof of case 1

In the second scenario when H_R has a vertex of red degree 4, say $v_{1,1}$, in order to avoid a blue C_4, we can assume that it is adjacent to three vertices of one partite set in red and one vertex of the other partite set in red. Without loss of generality assume that $v_{1,1}$ is adjacent in red to $v_{2,1}$, $v_{2,2}$, $v_{2,3}$ and $v_{3,1}$, and in blue to $v_{3,2}$ and $v_{3,3}$. But then as H_R has no red C_3, $(v_{3,1},v_{2,1})$, $(v_{3,1},v_{2,2})$ and $(v_{3,1},v_{2,3})$ have to be blue edges, as illustrated in Figure 2.

Next, in order to avoid a blue C_4, $v_{1,3}$ must be adjacent in red to two vertices of the second partite set, and to one of the vertices of $\{v_{3,2},v_{3,3}\}$. Without loss of generality assume that $(v_{1,3},v_{2,1})$, $(v_{1,3},v_{2,2})$ and $(v_{1,3},v_{3,3})$ are red. Then, in order to avoid a red C_3, $(v_{3,3}, v_{2,1})$ and $(v_{3,3}, v_{2,2})$ have to be blue. This forces $v_{3,3}\, v_{2,2}\, v_{3,1}\, v_{2,1}\, v_{3,3}$ to be a blue C_4, a contradiction.

Case 2: $j \in \{4,5,6\}$.
Consider the colouring of $K_{6\times 1} = H_R \oplus H_B$ generated by $H_B = 2K_3$. Then, $K_{6\times 1}$ has neither a red C_3 nor a blue C_4. This implies that $m_6(C_3,C_4) \geq 2$.

Claim 1: $m_4(C_3,C_4) \leq 2$.

Proof of claim 1: Consider any red/blue colouring given by $K_{4\times 2} = H_R \oplus H_B$, such that H_R contains no red C_3 and H_B contains no blue C_4. Suppose that there exist at least one vertex $v \in V(K_{4\times 2})$, such that it is incident to 4 or more red edges. Then as H_R contains no red C_3 all the edges of the induced subgraph generated by the set $N_R(v)$ will have to be blue. Hence, we obtain a blue C_4, a contradiction. Next, suppose that all the vertices of $v \in V(K_{4\times 2})$ are incident to at most 2 red edges. Then, each vertex of H_B has degree 4 or more, and by Bondy's lemma we obtain a blue C_4, a contradiction.

Now, we are left with the only possibility that there exists a vertex having a blue degree 3 and all the other vertices have a blue degree greater than or equal to 3. This gives rise to two possible cases.

Case a: The vertex of blue degree 3 is adjacent in blue, to vertices belonging to distinct partite sets.

Without loss of generality, assume $v_{1,2}$ is adjacent in red to $v_{2,2}$, $v_{3,2}$ and $v_{4,2}$. Since, H_R has no red C_3, $(v_{2,2},v_{3,2})$, $(v_{2,2},v_{4,2})$ and $(v_{3,2},v_{4,2})$ have to be blue. We know that the vertex $v_{1,1}$ must either be adjacent to two vertices of $\{v_{2,2},v_{3,2},v_{4,2}\}$ in blue or red. However, the first option is impossible, as it forces a blue C_4, and the second option must follow through. Therefore, we may assume that $(v_{1,1},v_{2,2})$ and $(v_{1,1},v_{3,2})$ are red. Also as $v_{4,1}$ doesn't lie in a blue C_4 we may assume that $(v_{4,1},v_{2,2})$ is also red. This is illustrated in Figure 3.

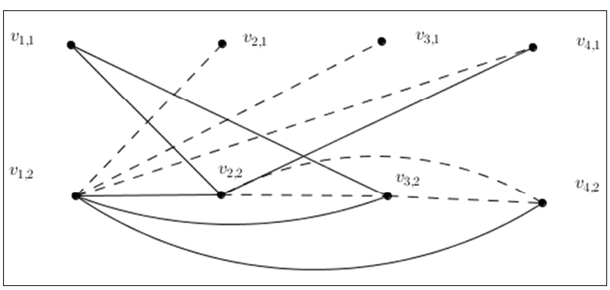

Figure 3: Graph used in proof of case a

As H_B has no blue C_4, $v_{1,1}$ must be adjacent to vertices $v_{2,1}$, $v_{3,1}$ in red. But then as there is no red C_3, these two vertices along with $v_{2,2}\, v_{3,2}$ will induce a blue C_4, a contradiction.

Case b: Every vertex of blue degree 3 is adjacent to two vertices u and v belonging to the same partite set V.

Remark: Given any vertex w of H_B, there exists a partite set V, such that w is adjacent in blue to u and v.

Without loss of generality, assume that $v_{1,2}$ is adjacent to $v_{2,1}$, $v_{2,2}$, $v_{3,2}$ in red and to $v_{3,1}$, $v_{4,1}$, $v_{4,2}$ in blue. But then by the above remark, in order to avoid a blue C_4 containing $v_{4,1}$ and $v_{4,2}$, both vertices $v_{2,1}$, $v_{2,2}$ must be adjacent to $v_{3,1}$, $v_{3,2}$ in blue. Then $v_{2,1}\, v_{3,1}\, v_{2,2}\, v_{3,2}\, v_{2,1}$ will be a blue C_4, a contradiction.

Theorem 5. *If* $j \geq 3$, *then*

$$m_j(C_3, K_{1,3}+x) = \begin{cases} \infty & if\ j \in \{3,4,5\}, \\ 2 & if\ j = 6, \\ 1 & if\ j \geq 7. \end{cases}$$

Proof. As a blue C_3 is a subgraph of a blue $K_{1,3}+x$, by theorem 3, we can conclude that $m_j(C_3,K_{1,3}+x) = \infty$ when $j \in \{3,4,5\}$.

Next consider the colouring of $K_{6\times 1} = H_R \oplus H_B$ where the edges $v_{11}v_{21}$, $v_{11}v_{31}$, $v_{21}v_{31}$, $v_{41}v_{51}$, $v_{41}v_{61}$ and $v_{51}v_{61}$ are blue and all the other edges are red (Figure 4). Then $K_{6\times 1}$ has neither a red C_3 nor a blue $K_{1,3}+x$. Therefore, $m_6(C_3,K_{1,3}+x) \geq 2$.

Now consider any red C_3-free colouring of $K_{6\times 2} = H_R \oplus H_B$.

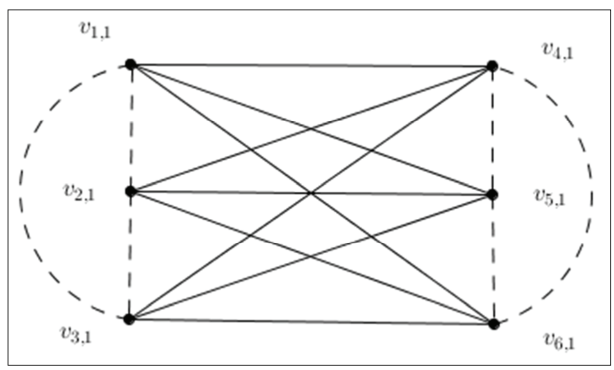

Figure 4: Graph used in proof of theorem 5

Then the induced subgraphs H_1 and H_2 of H_B generated by $\{v_{p1} \mid p \in \{1,...,6\}\}$ and $\{v_{p2} \mid p \in \{1,...,6\}\}$, respectively have a blue copy of C_3. Without loss of generality let the blue C_3 in H_1 be $v_{11}v_{21}v_{31}v_{11}$. Also let the blue C_3 in H_2 be $v_{x_1 2}\ v_{x_2 2}v_{x_3 2}\ v_{x_1 2}$ where x_1, x_2, $x_3 \in \{1,...,6\}$ and $x_1 < x_2 < x_3$.

If $x_3 = 3$ then one of the edges $v_{x_1 2}v_{31}$, $v_{x_1 2}v_{41}$, $v_{31}v_{41}$ is blue. Otherwise one of the edges v_{11}, $v_{x_2 2}$ $v_{x_2 2}v_{x_3 1}$, $v_{x_3 1}v_{11}$ is blue. Thus, in both situations, the graph has a blue $K_{1,3}+x$. Therefore, $m_6(C_3, K_{1,3}+x) = 2$.

Also, $m_j(C_3,K_{1,3}+x) = 1$ for $j \geq 7$ since $r(C_3,K_{1,3}+x) = 7$ (Chvátal & Harary, 1972). □

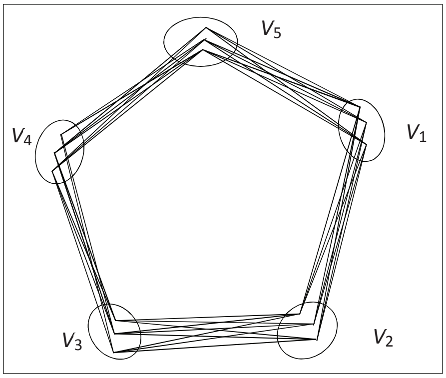

Figure 5: Graph used in proof of theorem 6

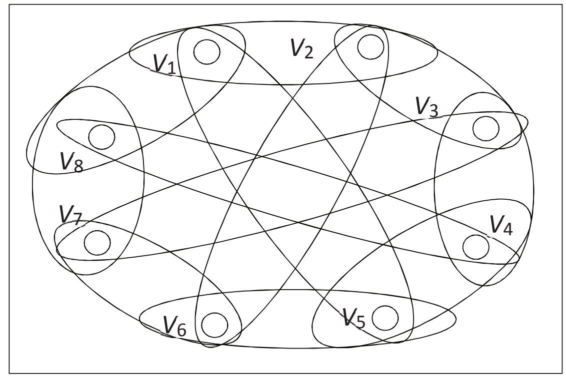

Figure 6: If two partite sets are contained in the same ellipse then all edges between them are coloured red

Theorem 6. *If* $j \geq 3$, *then*

$$m_j(C_3, B_2) = \begin{cases} \infty & if\ j \in \{3,4,5\}, \\ 2 & if\ j = 6, \\ 1 & if\ j \geq 7. \end{cases}$$

Proof. Let s be any positive integer. Consider the colouring of $K_{5\times s} = H_R \oplus H_B$ (Figure 5). Suppose that the five partite sets of $K_{5\times s}$ are $V_1, V_2, \ldots, V_5$

Given a partite set V_i, $i \in \{2,3,4\}$, colour all possible edges between V_i and its two neighbouring partite sets V_{i-1} and V_{i+1} in red. Also colour the edges between V_1 and V_5 in red. This is illustrated in Figure 5 for the case $s = 3$. Colour all the other edges in blue. Then the colouring of $K_{5\times s}$ contains no red C_3 or a blue B_2. This implies that $m_5(C_3,B_2) \geq$ s. Since s is arbitrary, we get that $m_5(C_3,B_2) = \infty$. Therefore, we conclude that $m_j(C_3,B_2) = \infty$ for all $j \leq 5$.

Next consider the colouring of $K_{6\times 1} = H_R \oplus H_B$, generated by $H_R = K_{3,3}$. Then, $K_{6\times 1}$ has neither a red C_3 nor a blue B_2. Hence $m_6(C_3,B_2) \geq 2$.

To prove that $m_6(C_3,B_2) \leq 2$, consider any red C_3-free and blue B_2-free colouring of $K_{6\times 2} = H_R \oplus H_B$. Since $r(C_3,C_3) = 6$, the induced subgraph H_1 of H_B generated by $\{v_{p1} \mid p \in \{1,\ldots,6\}\}$ will have a copy of a blue C_3. Without loss of generality let the blue C_3 in H_1 be $v_{11}v_{21}v_{31}v_{11}$. Let $H_2 = \{v_{41},v_{42},v_{52},v_{62}\}$. Then as H_2 does not induce a blue B_2 there exists two vertices x and y of H_2 adjacent to each other in red. Also as both x and y are not contained in a blue B_2, we see that both x and y are adjacent in red to at least two vertices of $V(H_1)$. However, this forces a red C_3, a contradiction. Therefore, $m_6(C_3,B_2) = 2$.

Clearly $m_j(C_3,B_2) = 1$ when $j \geq 7$ as $r(C_3,B_2) = 7$ (Chvátal & Harary, 1972).

Theorem 7. *If* $j \geq 3$, *then*

$$m_j(C_3, K_4) = \begin{cases} \infty & if\ j \in \{3,\ldots,8\}, \\ 1 & if\ j \geq 9. \end{cases}$$

Proof. Let s be any positive integer. Consider the colouring of $K_{8\times s} = H_R \oplus H_B$. Suppose that the partite sets of $K_{8\times s}$ are $V_1, V_2, \ldots, V_8$. Given a partite set V_i, $i \in \{2,3,\ldots,7\}$, colour all possible edges between V_i and its two neighbouring partite sets V_{i-1} and V_{i+1} in red. Also colour the edges between V_1 and V_8, V_1 and V_5, V_2 and V_6, V_3 and V_7, V_4 and V_8 in red. This is illustrated in Figure 6. Colour all the other edges in blue.

Then this colouring $K_{8\times s}$ contains neither a red C_3 nor a blue K_4. Therefore, we obtain that $m_8(C_3,K_4) \geq s$. Since s is arbitrary, we see that $m_8(C_3,K_4) = \infty$. Therefore, we conclude that $m_j(C_3,K_4) = \infty$ for all $j \leq 8$.

Clearly $m_j(C_3,K_4) = 1$ when $j \geq 9$ as $r(C_3,K_4) = 9$ (Chvátal & Harary, 1972).

$m_j(C_3,G)$ FOR ALL DISCONNECTED GRAPHS G UP TO 4 VERTICES

Let G be a disconnected graph with at most 4 vertices. By Theorem 1, we may assume that $G \neq 2K_2$. Let H be the connected graph on at most 3 vertices obtained by removing all the isolated vertices of G. Then we have $m_j(C_3,G) = m_j(C_3,H)$ unless $j = 3$ and $G = K_2 \cup 2K_1$, and this implies that $m_j(C_3,K_2 \cup 2K_1) = 2$. However, as we have found $m_j(C_3,H)$ for all possible connected graphs H on at most 3 vertices in the previous sections, we obtain $m_j(C_3,G)$ for all disconnected graphs G up to 4 vertices.

CONCLUSION

In conclusion, we have found all multipartite Ramsey numbers for a 3 cycle *versus* any graph up to 4 vertices.

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