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RESEARCH ARTICLE

Estimation of parameters of the 3-component mixture of Pareto distributions using type-I right censoring under Bayesian paradigm

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Abstract: As compared to simple probability models, a mixture model of some suitable lifetime distributions may be more capable of capturing the heterogeneity of nature. In this study, a 3-component mixture of Pareto distributions was investigated by considering the type-I right censoring scheme to obtain data from a heterogeneous population. First, considering a Bayesian structure, some mathematical properties of the 3-component mixture of Pareto distributions are discussed. These mathematical properties include Bayes estimators and posterior risks for the unknown component and proportion parameters using the uninformative (uniform and Jeffreys') and informative (gamma) priors under squared error loss and DeGroot loss functions. Then, the performance of the Bayes estimators for different sample sizes and test termination times under different loss functions were examined. In addition, limiting expressions of Bayes estimators and posterior risks are derived. Finally, the superiority of the Bayes estimators was established through a simulation study and a real life example.

Keywords: Bayesian estimation, censored data, Pareto distribution, posterior risk, uninformative and informative priors.

INTRODUCTION

In the current computational age, experts are able to explain estimates and predict and infer about the complicated structure of interest. In many practical studies, it is observed that the Pareto distribution can be used quite effectively in place of other lifetime distributions. Pareto distribution is often used for modelling many practical phenomena including city population sizes, incomes, sizes of firms and the lifetimes of certain objects. Abdel-All *et al*. (2003) presented the geometrical properties of the Pareto distribution and Ismail (2004) discussed a simple estimator for the shape parameter of the Pareto distribution. Sankaran and Nair (2005) studied the properties of finite mixture of Pareto distributions in income analysis perspective. Nadarajah and Kotz (2005) discussed the information matrix for a mixture of two Pareto distributions. The importance of Pareto distribution in modelling different real life phenomena is evident from the above mentioned studies.

In different practical applications, many types of data including simple and grouped data, censored data, progressively censored data and record values are analysed. Censoring is an important and valuable aspect of lifetime data. Due to time and cost issues, it is impossible to continue testing until the last observation. The values which are greater than the fixed life-test termination time are taken as censored observations. A valuable account on censoring have been given by Romeu (2004) and Gijbels (2010), and many others.

In recent years, finite mixtures of life distributions have proved to be of considerable interest both in terms of their methodological development and practical applications. Mixture models play a dynamic role in many real-life applications. As defined in Mendenhall and Hader (1958), for practical purposes the engineer may divide the failures of a system or a device into two or more different types of causes. In order to know the proportion of failure due to a certain cause and to improve the manufacturing process Acheson and McElwee (1952) divided electronic tube failures into gaseous defects, mechanical defects, and normal deterioration of the cathode.

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An engineering system is composed of different subsystems, which may be homogeneous or heterogeneous. Single probability models are not capable of capturing the heterogeneity of the nature of such systems. However, heterogeneity in the nature of such systems can be captured through mixture models.

Another important feature due to which finite mixture models of some suitable probability distributions receiving attention is, when a population is supposed to comprise a number of subpopulations mixed in an unknown proportion, then the common available distributions are irrelevant, e.g. a population of the lifetime of certain electrical elements or medicines may be divided into a number of subpopulations depending upon the possible cause of failure.

The use of mixture models in situations where data are given only from overall mixture distributions, is known as direct application of the mixture models. Direct applications of mixture models can be seen mostly in medicine, botany, zoology, paleoanthropology, agriculture, economics, life testing, reliability and survival analysis, etc. Li (1983), and Li and Sedransk (1988) discussed different features of mixture models and defined two types of mixture models, namely, type-I and type-II mixture models. The mixture of probability density functions from the same (different) family is known as type-I (type-II) mixture model. Mixture models have been successfully applied in many areas such as engineering, physical sciences, chemical sciences, biological sciences, etc. Several authors have applied mixture modelling in different practical problems. For example, Harris (1983) fitted mixture distributions to model crime and justice data. Kanji (1985) described wind shear data using mixture distributions, while Jones and McLachlan (1990) applied a mixture of normal distribution and Laplace distribution to model wind shear data.

Most of the researchers focused on Bayesian and classical analysis of 2-component mixture models. Rider (1961) used the method of moments to obtain the estimates of parameters of a mixture of two exponential distributions. Sinha (1998) used the Bayesian approach to estimate the parameters of the 2-component mixture model considered by Mendenhall and Hader (1958). Sultan *et al*. (2007) investigated the properties of the 2-component mixture of inverse Weibull distributions. Saleem and Aslam (2009) discussed the use of informative and non-informative priors for Bayesian analysis of a 2-component mixture of the Rayleigh distribution. Also, Saleem *et al*. (2010) presented the Bayesian analysis of a 2-component mixture of power distribution using complete and censored samples. Kazmi *et al*. (2012) described the Bayesian analysis for a 2-component mixture of the Maxwell distributions. Ali *et al*. (2013) analysed comparisons of the informative priors for the scale parameter of mixture of the Laplace distribution under different loss functions. Feroze and Aslam (2014) presented the Bayesian estimation procedure for analysing the lifetime data under doubly censored sampling using a 2-component mixture of the Weibull distribution.

Motivated by the above mentioned applications of mixture models, in the current study, we studied Bayesian estimation of parameters of a 3-component mixture of the Pareto distributions. All the parameters of a mixture of distributions are assumed unknown. Bayesian analysis was performed by considering different priors and loss functions using direct application of mixture models. In addition, an ordinary type-I right censoring was also applied.

METHODOLOGY

3-Component mixture of Pareto distributions

A random variable Y is said to follow a finite mixture distribution with q components if the density function of Y can be written in the form: $f(y)=\sum_{m=1}^{q} p_m f_m(y)$, where p_m $(m=1,\ 2,...,\ q)$ is the m^{th} mixing proportion such that $p_q = 1-\sum_{m=1}^{q-1} p_m$ and $f_m(y)$ is m^{th} component density function. A finite 3-component mixture of Pareto distributions with mixing proportions p_1, p_2 and $(1-p_1-p_2)$ has its pdf as:

$$f(y;\ \mathbf{\Phi}) = p_1 f_1(y;\ \Phi_1) + p_2 f_2(y;\ \Phi_2) + (1-p_1-p_2) f_3(y;\ \Phi_3),\ p_1,\ p_2 \ge 0,\ p_1+p_2 \le 1, \quad ...(1)$$

where $\mathbf{\Phi} = (\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1,\ p_2)$, $\Phi_m = \lambda_m$, $m=1,\ 2,\ 3$ and $f_m(y;\ \Phi_m)$, the pdf of the m^{th} component, is written as:

$$f_m(y;\ \Phi_m) = \lambda_m y^{-(\lambda_m+1)},\ 1<y<\infty,\ \lambda_m>0,\ m=1,\ 2,\ 3. \quad ...(2)$$

The cdf of a 3-component mixture of Pareto distributions is:

$$F(y;\ \mathbf{\Phi}) = p_1 F_1(y;\ \Phi_1) + p_2 F_2(y;\ \Phi_2) + (1-p_1-p_2) F_3(y;\ \Phi_3), \quad ...(3)$$

where $F_m\left(y;\ \Phi_m\right)$, the cdf of the m^{th} component, is given by:

$$F_m\left(y;\ \Phi_m\right)=1-y^{-\lambda_m},\ 1<y<\infty,\ \lambda_m>0,\ m=1,\ 2,\ 3. \quad \text{...(4)}$$

The likelihood function

Supposing that n units are used in a life testing experiment with a fixed test termination time t, and lifetime of the units follows a 3-component mixture of Pareto distributions, the experiment is performed and it is observed that r units out of n units failed until fixed test termination time t and remaining $n-r$ units are still functioning. It may be pointed out that out of r failures, r_1, r_2 and r_3 failures belong to subpopulation-I, subpopulation-II and subpopulation-III, respectively, depending upon the reasons of failure. So the number of uncensored observations is $r=r_1+r_2+r_3$ and the remaining $n-r$ observations are censored that give no information as to which subpopulation they belong. We define y_{lk}, $0<y_{lk}\le t$, be the failure time of the k^{th} $\left(k=1,\ 2,\cdots,\ r_l\right)$ unit belonging to the $l^{\text{th}}\left(l=1,\ 2,\ 3\right)$ sub population. The likelihood function for 3-component mixture model can be written as:

$$L\left(\mathbf{\Phi}|\mathbf{y}\right)\propto\left\{\prod_{k=1}^{r_1}p_1f_1\left(y_{1k}\right)\right\}\left\{\prod_{k=1}^{r_2}p_2f_2\left(y_{2k}\right)\right\}\left\{\prod_{k=1}^{r_3}\left(1-p_1-p_2\right)f_3\left(y_{3k}\right)\right\}\left\{1-F\left(t\right)\right\}^{n-r}. \quad \text{...(5)}$$

After simplification, the likelihood function of the 3-component mixture of Pareto distributions becomes:

$$L\left(\mathbf{\Phi}|\mathbf{y}\right)\propto\left[\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp\left\{-\lambda_1\left(\left(n-r-i\right)\ln t+\sum_{k=1}^{r_1}\ln y_{1k}\right)\right\}\exp\left\{-\lambda_2\left(\left(i-j\right)\ln t+\sum_{k=1}^{r_2}\ln y_{2k}\right)\right\}\right.$$

$$\left.\exp\left\{-\lambda_3\left(\left(j\right)\ln t+\sum_{k=1}^{r_3}\ln y_{3k}\right)\right\}\lambda_1^{r_1}\lambda_2^{r_2}\lambda_3^{r_3}\,p_1^{n-r-i+r_1}\,p_2^{i-j+r_2}\left(1-p_1-p_2\right)^{j+r_3}\right], \quad \text{...(6)}$$

where

$$\mathbf{y}=\left(y_{11},\ y_{12},...,\ y_{1r_1},\ y_{21},\ y_{22},...,\ y_{2r_2},\ y_{31},\ y_{32},...,\ y_{3r_3}\right).$$

Situations exist where no prior information on the parameter of interest is available. In such situations, one has to use an uninformative prior distribution. Jeffreys (1946) suggested a method based on the square-root of the Fisher information to determine an uninformative prior. Later on, Geisser (1984) proposed some techniques to determine an uninformative prior. Bernardo (1979) argued that an uninformative prior should be regarded as a reference prior, i.e., a prior that is convenient to use as a standard when analysing statistical data. The most commonly used uninformative priors are the uniform prior (UP) and the Jeffreys' prior (JP). Both priors are used only when no formal prior information is available. That is why, in this study, we assumed the uniform and Jeffreys' priors as the prior distributions.

On the other hand, when definite information is available, it is quantified as an informative prior. Using an informative prior along with the sample information is usually thought of as updating the current information, which helps reducing the posterior risks of the Bayes estimators. It was assumed that the availability of prior information on component parameters as a gamma distribution while a bivariate beta prior distribution is assumed for mixing proportions.

Posterior distribution using the uninformative priors

In this study, it was assumed that the improper UP which is proportional to a constant for the unknown component parameter λ_m, i.e., $\lambda_m\sim Uniform\left(0,\ \infty\right)$, $m=1,\ 1,\ 2,$ and the UP over the interval $\left(0,\ 1\right)$ for the unknown proportion parameter p_s, i.e., $p_s\sim Uniform\left(0,\ 1\right)$, $s=1,\ 2$. Assuming the independence of parameters, the joint prior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 is $\pi_1\left(\mathbf{\Phi}\right)\propto 1$.

The JP was defined as $p\left(\lambda_m\right)\propto\sqrt{\left|I\left(\lambda_m\right)\right|}$, $m=1,\ 2,\ 3,$ where $I\left(\lambda_m\right)=-E\left[\frac{\partial^2 f\left(y;\lambda_m\right)}{\partial\lambda_m^2}\right]$ is the Fisher's information matrix. The prior distributions of the proportion parameters p_1 and p_2 were again assumed to be uniform over the interval $\left(0,\ 1\right)$, i.e., $p_s\sim Uniform\left(0,\ 1\right)$, $s=1,\ 2$. Assuming the independence of parameters, the joint prior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 is $\pi_2\left(\mathbf{\Phi}\right)\propto\frac{1}{\lambda_1\lambda_2\lambda_3}$.

The joint posterior distribution of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 given data $\mathbf{y}$ assuming the UP and the JP is as follows:

$$g_w\left(\boldsymbol{\Phi}|\mathbf{y}\right)=\frac{L\left(\boldsymbol{\Phi}|\mathbf{y}\right)\pi_w\left(\boldsymbol{\Phi}\right)}{\int_{\boldsymbol{\Phi}}L\left(\boldsymbol{\Phi}|\mathbf{y}\right)\pi_w\left(\boldsymbol{\Phi}\right)d\boldsymbol{\Phi}}, \qquad ...(7)$$

$$g_w\left(\boldsymbol{\Phi}|\mathbf{y}\right)=\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp\left(-B_{1w}\lambda_1\right)\exp\left(-B_{2w}\lambda_2\right)\exp\left(-B_{3w}\lambda_3\right)p_1^{C_{0w}-1}p_2^{D_{0w}-1}\left(1-p_1-p_2\right)^{E_{0w}-1}}{H_w\,\lambda_1^{1-A_{1w}}\lambda_2^{1-A_{2w}}\lambda_3^{1-A_{3w}}} \qquad ...(8)$$

where $A_{11}=r_1+1$, $A_{21}=r_2+1$, $A_{31}=r_3+1$, $A_{12}=r_1$, $A_{22}=r_2$, $A_{32}=r_3$, $B_{1w}=\left(n-r-i\right)\ln t+\sum_{k=1}^{r_1}\ln y_{1k}$,

$B_{2w}=\left(i-j\right)\ln t+\sum_{k=1}^{r_2}\ln y_{2k}$,

$B_{3w}=\left(j\right)\ln t+\sum_{k=1}^{r_3}\ln y_{3k}$, $C_{0w}=n-r-i+r_1+1$, $D_{0w}=i-j+r_2+1$, $E_{0w}=j+r_3+1$,

$H_w=\Gamma\left(A_{1w}\right)\Gamma\left(A_{2w}\right)\Gamma\left(A_{3w}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1w}^{-A_{1w}}B_{2w}^{-A_{2w}}B_{3w}^{-A_{3w}}B\left(C_{0w},D_{0w},E_{0w}\right)$.

where $w=1$ for the UP and $w=2$ for the JP.

Posterior distribution using the informative prior

As an informative prior (IP) distribution, gamma distributions were assumed to be the prior distributions for component parameters λ_1, λ_2, λ_3 and the bivariate beta distribution was assumed to be the prior distribution for proportion parameters p_1, p_2. Symbolically, we have:

$$\pi_4\left(\lambda_m;\ a_m,\ b_m\right)=\frac{b_m^{a_m}}{\Gamma\left(a_m\right)}\lambda_m^{a_m-1}\exp\left(-b_m\lambda_m\right),$$
$$m=1,\ 2,\ 3,\ \lambda_m>0,\ a_m,b_m>0, \qquad ...(9)$$

$$\pi_5\left(p_1,\ p_2;\ a,\ b,\ c\right)=\frac{1}{B\left(a,b,c\right)}p_1^{a-1}p_2^{b-1}\left(1-p_1-p_2\right)^{c-1},$$
$$p_1,p_2\geq 0,\ p_1+p_2\leq 1,\ a,b,c>0. \qquad ...(10)$$

The joint prior distribution of parameters λ_1, λ_2, λ_3, p_1 and p_2 using the IP is:

$$\pi_3\left(\boldsymbol{\Phi}\right)=\frac{b_1^{a_1}}{\Gamma\left(a_1\right)}\lambda_1^{a_1-1}\exp\left(-b_1\lambda_1\right)\frac{b_2^{a_2}}{\Gamma\left(a_2\right)}\lambda_2^{a_2-1}\exp\left(-b_2\lambda_2\right)$$
$$\frac{1}{B\left(a,b,c\right)}p_1^{a-1}p_2^{b-1}\left(1-p_1-p_2\right)^{c-1}. \qquad ...(11)$$

The joint posterior distribution of parameters λ_1, λ_2, λ_3, p_1 and p_2 given data **y**, using the IP is:

$$g_3\left(\boldsymbol{\Phi}|\mathbf{y}\right)=\frac{L\left(\boldsymbol{\Phi}|\mathbf{y}\right)\pi_3\left(\boldsymbol{\Phi}\right)}{\int_{\boldsymbol{\Phi}}L\left(\boldsymbol{\Phi}|\mathbf{y}\right)\pi_3\left(\boldsymbol{\Phi}\right)d\boldsymbol{\Phi}}, \qquad ...(12)$$

$$g_3\left(\boldsymbol{\Phi}|\mathbf{y}\right)=\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp\left(-B_{13}\lambda_1\right)\exp\left(-B_{23}\lambda_2\right)\exp\left(-B_{33}\lambda_3\right)p_1^{C_{03}-1}p_2^{D_{03}-1}\left(1-p_1-p_2\right)^{E_{03}-1}}{H_3\,\lambda_1^{1-A_{13}}\lambda_2^{1-A_{23}}\lambda_3^{1-A_{33}}} \qquad ...(13)$$

where $A_{13}=r_1+a_1$, $A_{23}=r_2+a_2$, $A_{33}=r_3+a_3$,

$B_{13}=\left(n-r-i\right)\ln t+\sum_{k=1}^{r_1}\ln y_{1k}+b_1$,

$B_{23}=\left(i-j\right)\ln t+\sum_{k=1}^{r_2}\ln y_{2k}+b_2$,

$B_{33}=\left(j\right)\ln t+\sum_{k=1}^{r_3}\ln y_{3k}+b_3$,

$C_{03}=n-r-i+r_1+a$,

$D_{03}=i-j+r_2+b$, $E_{03}=j+r_3+c$,

$H_3=\Gamma\left(A_{13}\right)\Gamma\left(A_{23}\right)\Gamma\left(A_{33}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}$

$B_{33}^{-A_{33}}B\left(C_{03},D_{03},E_{03}\right)$.

Marginal posterior distributions

The marginal posterior distributions of parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 using the UP, the JP and the IP given data $\mathbf{y}$ are:

$$h_v\left(\lambda_\varpi|\mathbf{y}\right)=\int_{p_2}\int_{p_1}\int_{\lambda_\eta}\int_{\lambda_\pi} g_v\left(\mathbf{\Phi}|\mathbf{y}\right)d\lambda_\pi d\lambda_\eta dp_1 dp_2\ , \qquad ...(14)$$

$$h_v\left(\lambda_\varpi|\mathbf{y}\right)=\frac{\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)}{H_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}$$

$$B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right)\lambda_\varpi^{A_{\varpi v}-1}$$

$$\exp\left(-B_{\varpi v}\lambda_\varpi\right),\ \lambda_\varpi>0, \qquad ...(15)$$

where ϖ,π and η take the values as: (i) $\varpi=1,\ \pi=2,\ \eta=3$ (ii) $\varpi=2,\ \pi=1,\ \eta=3$ and (iii) $\varpi=3,\ \pi=1,\ \eta=2$. Also, $B(\cdot\ ,\ \cdot)$ is the usual beta function.

$$h_v\left(p_\xi|\mathbf{y}\right)=\int_{p_\varepsilon}\int_{\lambda_3}\int_{\lambda_2}\int_{\lambda_1} g_v\left(\mathbf{\Phi}|\mathbf{y}\right)d\lambda_1 d\lambda_2 d\lambda_3 dp_\varepsilon\ , \qquad ...(16)$$

$$h_v\left(p_\xi|\mathbf{y}\right)=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)}{H_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}$$

$$B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)p_\xi^{\Delta_{0v}-1}\left(1-p_\xi\right)^{\Upsilon_{0v}+E_{0v}-1},0<p_\xi<1, \qquad ...(17)$$

where ξ and ε take the values as: (i) $\xi=1,\ \varepsilon=2,\ \Upsilon=D,\ \Delta=C$ and (ii) $\xi=2,\ \varepsilon=1,\ \Upsilon=C,\ \Delta=D$. Also, $v=1$ for the UP, $v=2$ for the JP and $v=3$ for the IP.

Bayesian estimation under loss functions

In this section, we focus on the derivation of the Bayes estimators and posterior risks using the UP, the JP and the IP under squared error loss function (SELF) and DeGroot loss function (DLF). The Bayes decision is a decision which reduces the risk function which is then known as the best decision. If $L(\lambda,\ d)$ is a loss function then the expected value of the loss function for a given decision with respect to the posterior distribution is the posterior risk function, and if $\hat{d}$ is a Bayes estimator then $\rho\left(\hat{d}\right)$ is called the posterior risk and is defined as: $\rho\left(\hat{d}\right)=E_{\lambda|\mathbf{y}}\left\{L\left(\lambda,\ \hat{d}\right)\right\}$. Our purpose in this study was to look for better Bayes estimators of the different considered parameters. That is the reason considering different loss functions and searching for minimum posterior risk. Two different loss functions, namely the SELF and the DLF were used to obtain the Bayes estimators and their posterior risks.

Bayes estimators and posterior risks under SELF: The SELF $L(\lambda,\ d)=(\lambda-d)^2$ was proposed by Legendre (1806) to develop the least square theory. By using SELF the Bayes estimators and posterior risks are given by $\hat{d}=E_{\lambda|\mathbf{y}}\left(\lambda\right)$ and $\rho\left(\hat{d}\right)=E_{\lambda|\mathbf{y}}\left(\lambda^2\right)-\left\{E_{\lambda|\mathbf{y}}\left(\lambda\right)\right\}^2$, respectively. The respective marginal posterior distributions yield the Bayes estimators and posterior risks assuming the UP, the JP and the IP for parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ p_1$ and p_2 under SELF as:

$$\hat{\lambda}_{\varpi v}=\frac{\Gamma\left(A_{\varpi v}+1\right)\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)}{H_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$B_{\varpi v}^{-(A_{\varpi v}+1)}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right) \qquad ...(18)$$

$$\hat{p}_{\xi v}=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)}{H_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)B\left(\Delta_{0v}+1,\Upsilon_{0v}+E_{0v}\right) \qquad ...(19)$$

$$\rho\left(\hat{\lambda}_{\varpi v}\right)=\frac{\Gamma\left(A_{\varpi v}+2\right)\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)}{H_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$B_{\varpi v}^{-(A_{\varpi v}+2)}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right)-\left\{\hat{\lambda}_{\varpi v}\right\}^2 \qquad ...(20)$$

$$\rho\left(\hat{p}_{\xi v}\right)=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)}{H_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)B\left(\Delta_{0v}+2,\Upsilon_{0v}+E_{0v}\right)-\left\{\hat{p}_{\xi v}\right\}^2 \qquad ...(21)$$

Bayes estimators and posterior risks under DLF: The DLF suggested by DeGroot (2005) is defined as $L(\lambda,\ d)=\left(\frac{\lambda-d}{d}\right)^2$. The Bayes estimators and their posterior risk under DLF are $\hat{d}=\frac{E_{\lambda|\mathbf{y}}\left(\lambda^2\right)}{E_{\lambda|\mathbf{y}}\left(\lambda\right)}$ and $\rho\left(\hat{d}\right)=1-\frac{\left\{E_{\lambda|\mathbf{y}}\left(\lambda\right)\right\}^2}{E_{\lambda|\mathbf{y}}\left(\lambda^2\right)}$, respectively. So the Bayes estimators and posterior risks using the UP, the JP and the IP for parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ P_1$ and P_2 under DLF are obtained as:

$$\hat{\lambda}_{\varpi v}=\frac{\Gamma\left(A_{\varpi v}+2\right)\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{\varpi v}^{-\left(A_{\varpi v}+2\right)}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right)}{\Gamma\left(A_{\varpi v}+1\right)\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{\varpi v}^{-\left(A_{\varpi v}+1\right)}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right)} \quad ...(22)$$

$$\hat{p}_{\xi v}=\frac{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)B\left(\Delta_{0v}+2,\Upsilon_{0v}+E_{0v}\right)}{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)B\left(\Delta_{0v}+1,\Upsilon_{0v}+E_{0v}\right)} \quad ...(23)$$

$$\rho\left(\hat{\lambda}_{\varpi v}\right)=1-\frac{\left\{\Gamma\left(A_{\varpi v}+1\right)\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{\varpi v}^{-\left(A_{\varpi v}+1\right)}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right)\right\}^{2}}{H_{v}\Gamma\left(A_{\varpi v}+2\right)\Gamma\left(A_{\pi v}\right)\Gamma\left(A_{\eta v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{\varpi v}^{-\left(A_{\varpi v}+2\right)}B_{\pi v}^{-A_{\pi v}}B_{\eta v}^{-A_{\eta v}}B\left(C_{0v},E_{0v}\right)B\left(D_{0v},C_{0v}+E_{0v}\right)} \quad ...(24)$$

$$\rho\left(\hat{p}_{\xi v}\right)=1-\frac{\left\{\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)B\left(\Delta_{0v}+1,\Upsilon_{0v}+E_{02}\right)\right\}^{2}}{H_{v}\Gamma\left(A_{1v}\right)\Gamma\left(A_{2v}\right)\Gamma\left(A_{3v}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B\left(\Upsilon_{0v},E_{0v}\right)B\left(\Delta_{0v}+2,\Upsilon_{0v}+E_{02}\right)} \quad ...(25)$$

Elicitation of hyperparameters

Elicitation is a process used to quantify a person's professional belief and knowledge about the subject matter. In Bayesian perspective, elicitation most often arises as a method of specifying the prior distribution of the random parameter(s). Elicitation is simply the quantification of prior knowledge about the random parameter(s) so that this can then be combined with the likelihood to obtain posterior distribution for further statistical analysis. In this study, we adopted the prior predictive method based on predictive probabilities suggested by Aslam (2003). For eliciting the hyperparameters, prior predictive distribution (PPD) was used. The PPD using the IP for a random variable Y is defined as:

$$p\left(y\right)=\int_{\Phi}f\left(y|\Phi\right)\pi_{3}\left(\Phi\right)d\Phi \quad ...(26)$$

On substituting equations (1) and (11) in equation (26) and then simplifying, we get:

$$p\left(y\right)=\frac{1}{\left(a+b+c\right)y}\left[\frac{aa_{1}b_{1}^{a_{1}}}{\left(b_{1}+\ln y\right)^{a_{1}+1}}+\frac{ba_{2}b_{2}^{a_{2}}}{\left(b_{2}+\ln y\right)^{a_{2}+1}}+\frac{ca_{3}b_{3}^{a_{3}}}{\left(b_{3}+\ln y\right)^{a_{3}+1}}\right]. \quad ...(27)$$

Using the prior predictive distribution given in equation (27), we consider nine intervals (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9) and (9, 10) with respective probabilities 0.45, 0.10, 0.05, 0.03, 0.025, 0.02, 0.015, 0.01 and 0.008 as an expert's belief about these intervals. Using equation (27), the following nine equations in (28) are solved simultaneously in Mathematica package for eliciting the hyper parameters a_1, b_1, a_2, b_2, a_3, b_3, a, b and c.

$$\int_{1}^{2}p\left(y\right)dy=0.45;\quad \int_{2}^{3}p\left(y\right)dy=0.10;\quad \int_{3}^{4}p\left(y\right)dy=0.05;$$
$$\int_{4}^{5}p\left(y\right)dy=0.03;\quad \int_{5}^{6}p\left(y\right)dy=0.025;\quad \int_{6}^{7}p\left(y\right)dy=0.02;$$
$$\int_{7}^{8}p\left(y\right)dy=0.015;\quad \int_{8}^{9}p\left(y\right)dy=0.01;\quad \int_{9}^{10}p\left(y\right)dy=0.008; \quad ...(28)$$

The elicited values of the hyper parameters a_1, b_1, a_2, b_2, a_3, b_3, a, b and c were obtained as 0.9295, 0.8154, 0.7755, 0.6344, 0.573, 0.4377, 2.315, 2.484 and 2.072, respectively.

Limiting expressions for complete data set

When the test termination time t tends to ∞, uncensored observations r tends to sample size n and r_l tends to $n_l(l=1,2,3)$, so that all the observations which are censored became uncensored in our analysis. So the information contained in the sample is increased and consequently the posterior risks of the Bayes estimators reduced. The efficiency of the Bayes estimators is increased because all the observations are incorporated in our sample. The limiting expressions for Bayes estimators and posterior risks using the UP, the JP and the IP under SELF and DLF are given in Tables 1 to 4.

Table 1: Limiting expressions for Bayes estimators as $t \to \infty$ under SELF

Bayes estimators	Bayes estimators		
	UP	JP	IP
$\hat{\lambda}_1$	$\frac{n_1+1}{\sum_{k=1}^{n_1}\ln y_{1k}}$	$\frac{n_1}{\sum_{k=1}^{n_1}\ln y_{1k}}$	$\frac{n_1+a_1}{\sum_{k=1}^{n_1}\ln y_{1k}+b_1}$
$\hat{\lambda}_2$	$\frac{n_2+1}{\sum_{k=1}^{n_2}\ln y_{2k}}$	$\frac{n_2}{\sum_{k=1}^{n_2}\ln y_{2k}}$	$\frac{n_2+a_2}{\sum_{k=1}^{n_2}\ln y_{2k}+b_2}$
$\hat{\lambda}_3$	$\frac{n_3+1}{\sum_{k=1}^{n_3}\ln y_{3k}}$	$\frac{n_3}{\sum_{k=1}^{n_3}\ln y_{3k}}$	$\frac{n_3+a_3}{\sum_{k=1}^{n_3}\ln y_{3k}+b_3}$
$\hat{p}_1$	$\frac{n_1+1}{n+3}$	$\frac{n_1+1}{n+3}$	$\frac{n_1+a}{n+a+b+c}$
$\hat{p}_2$	$\frac{n_2+1}{n+3}$	$\frac{n_2+1}{n+3}$	$\frac{n_2+b}{n+a+b+c}$

Table 2: Limiting expressions for posterior risks as $t \to \infty$ under SELF

Posterior risks	Posterior risks		
	UP	JP	IP
$\rho(\hat{\lambda}_1)$	$\frac{n_1+1}{\left(\sum_{k=1}^{n_1}\ln y_{1k}\right)^2}$	$\frac{n_1}{\left(\sum_{k=1}^{n_1}\ln y_{1k}\right)^2}$	$\frac{n_1+a_1}{\left(\sum_{k=1}^{n_1}\ln y_{1k}+b_1\right)^2}$
$\rho(\hat{\lambda}_2)$	$\frac{n_2+1}{\left(\sum_{k=1}^{n_2}\ln y_{2k}\right)^2}$	$\frac{n_2}{\left(\sum_{k=1}^{n_2}\ln y_{2k}\right)^2}$	$\frac{n_2+a_2}{\left(\sum_{k=1}^{n_2}\ln y_{2k}+b_2\right)^2}$
$\rho(\hat{\lambda}_3)$	$\frac{n_3+1}{\left(\sum_{k=1}^{n_3}\ln y_{3k}\right)^2}$	$\frac{n_3}{\left(\sum_{k=1}^{n_3}\ln y_{3k}\right)^2}$	$\frac{n_3+a_3}{\left(\sum_{k=1}^{n_3}\ln y_{3k}+b_3\right)^2}$
$\rho(\hat{p}_1)$	$\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_1+a)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$
$\rho(\hat{p}_2)$	$\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_2+b)(n_1+n_3+a+c)}{(n+a+b+c)^2(n+a+b+c+1)}$

Table 3: Limiting expressions for Bayes estimators as under DLF

Bayes estimators	Bayes estimators		
	UP	JP	IP
$\hat{\lambda}_1$	$\frac{n_1+2}{\sum_{k=1}^{n_1}\ln y_{1k}}$	$\frac{n_1+1}{\sum_{k=1}^{n_1}\ln y_{1k}}$	$\frac{n_1+a_1+1}{\sum_{k=1}^{n_1}y_{1k}+b_1}$
$\hat{\lambda}_2$	$\frac{n_2+2}{\sum_{k=1}^{n_2}\ln y_{2k}}$	$\frac{n_2+1}{\sum_{k=1}^{n_2}\ln y_{2k}}$	$\frac{n_2+a_2+1}{\sum_{k=1}^{n_2}y_{2k}+b_2}$
$\hat{\lambda}_3$	$\frac{n_3+2}{\sum_{k=1}^{n_3}\ln y_{3k}}$	$\frac{n_3+1}{\sum_{k=1}^{n_3}\ln y_{3k}}$	$\frac{n_3+a_3+1}{\sum_{k=1}^{n_3}y_{3k}+b_3}$
$\hat{p}_1$	$\frac{n_1+2}{n+4}$	$\frac{n_1+2}{n+4}$	$\frac{n_1+a+1}{n+a+b+c+1}$
$\hat{p}_2$	$\frac{n_2+2}{n+4}$	$\frac{n_2+2}{n+4}$	$\frac{n_2+b+1}{n+a+b+c+1}$

Table 4: Limiting expressions for Posterior risks as under DLF

Posterior risks	Posterior risks		
	UP	JP	IP
$\rho(\hat{\lambda}_1)$	$\frac{1}{n_1+2}$	$\frac{1}{n_1+1}$	$\frac{1}{n_1+a_1+1}$
$\rho(\hat{\lambda}_2)$	$\frac{1}{n_2+2}$	$\frac{1}{n_2+1}$	$\frac{1}{n_2+a_2+1}$
$\rho(\hat{\lambda}_3)$	$\frac{1}{n_3+2}$	$\frac{1}{n_3+1}$	$\frac{1}{n_3+a_3+1}$
$\rho(\hat{p}_1)$	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+b+c)}{(n_1+a+1)(n+a+b+c)}$
$\rho(\hat{p}_2)$	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+a+c)}{(n_2+b+1)(n+a+b+c)}$

Simulation study

A simulation study was carried out to scrutinise the performance of the Bayes estimators under different priors, loss functions, parametric values, sample sizes and test termination times. A random sample of fixed size *n* was taken for the 3-component mixture of Pareto distributions with different combinations of parameters $\left(\lambda_1, \lambda_2, \lambda_3, p_1, p_2\right)$ at a fixed test termination time. To generate a mixture of data, we used probabilistic mixing proportions p_1 and p_2. The $p_1 n$ observations were selected randomly from the first component density $f_1\left(y;\ \lambda_1\right)$, $p_2 n$ observations were taken randomly from second component density $f_2\left(y;\ \lambda_2\right)$ and the remaining $\left(1-p_1-p_2\right)n$ observations were chosen randomly from the third component density $f_3\left(y;\ \lambda_3\right)$. A sample censored at a fixed test termination time t was selected in order to evaluate the impact of the test termination time on Bayes estimators. The observations which are greater than a fixed test termination time t were taken as censored ones. The choice of the test termination time is made in such a way that the censoring rate in the resulting sample is approximately 10 % to 25 %. On the basis of the generated sample, Bayes estimates and posterior risks are computed through Mathematica package.The whole procedure is iterated 1000 times. The simulated results are then averaged over 1000 values. The simulated Bayes estimates and posterior risks using the UP, JP and IP under SELF and DLF are showcased in the Tables I – VI (Appendix).

RESULTS

From Tables I – VI (Appendix), it is observed that the extent of under-estimation (over-estimation) of the component and proportion parameters (through Bayes estimators) using the UP, JP and IP under SELF and DLF is smaller for larger test termination time (sample size) as compared to smaller test termination time (sample size) at different sample sizes (test termination times). Similarly, the extent of over-estimation (under-estimation) of the component and proportion parameters is greater for smaller values of component parameters as compared to larger values of component parameters at different test termination times and sample sizes. Also, the difference of the Bayes estimates from the assumed parameters reduce to zero with an increase in the sample size at different fixed test termination times, and the same is the case with larger test termination time as compared to small test termination time for varying sample sizes.

It is also observed that the posterior risks of Bayes estimators using the UP, JP and IP under SELF and DLF are reduced with an increase in sample size at different test termination times. For a smaller test termination time, the posterior risks of Bayes estimators are larger than the posterior risks for large test termination time irrespective of the prior, loss function and sample size. Also, the posterior risks of Bayes estimators of component parameters are smaller (larger) for smaller component parametric values under SELF (DLF) for each sample size and test termination time considered in the simulation study. However, the posterior risks of Bayes estimators of proportion parameters are larger for smaller component parametric values under SELF and DLF for each sample size and test termination time.

As far as the problem of selecting a suitable prior is concerned, it can be seen that having the smallest associated posterior risk for a given loss function IP emerges as the best prior amongst the different uninformative and informative priors considered in this study. On the other hand, the DLF is observed performing better than SELF for estimating the component parameters, whereas for estimating the proportion parameters, SELF is observed superior to DLF. It should be noted that the selection of the best prior (loss function) for a given loss function (prior) is made based on the posterior risks associated with it. Also, the selection of the best prior and loss function does not depend on the sample size and test termination time.

A real life example

Davis (1952) reported a mixture data, x = (x_{11}, x_{12},..., x_{1r_1}, x_{21}, x_{22},..., x_{2r_2}, x_{31}, x_{32},..., x_{3r_3}) on lifetimes (in thousand hours) of many components used in aircraft sets. To illustrate the proposed methodology, we take the data on three components, namely, V805 Transmitter Tube**,** Transmitter Tube and V600 Indicator Tube. Davis showed that data x can be modelled by a mixture of exponential distributions. The transformation $y = \exp(x)$ of an exponential random data (x) yields the Pareto random data (y). This transformation allows us to use the Davis mixture data for applying the proposed Bayesian analysis. It is unknown as to which component fails until a failure (of a radar set) occurs at or before the test termination time 0.6 hour. The total number of tests is conducted 1340 times. The data summary required to evaluate the Bayes estimates and posterior risks is given by:

$$\sum_{k=1}^{r_1}\ln\left(y_{1k}\right)=134.080,\ \sum_{k=1}^{r_2}\ln\left(y_{2k}\right)=50.375,\ \sum_{k=1}^{r_3}\ln\left(y_{3k}\right)=16.250,\ n=1340,\ r_1=866,\ r_2=337,\ r_3=83,\ r=1286.$$

Since $n-r=54$, we have almost 5 % censored sample. Thus, this is a type-I right censored data. Bayes estimates

Table 5: Bayes estimates (BE) and posterior risks (PR) using the UP, the JP and the IP under SELF and DLF with Davis (1952) real life mixture data

Prior	Loss function		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
UP	SELF	BE	5.66727881	5.98128700	3.64703769	0.66911626	0.25940614
		PR	**0.05190606**	**0.16525949**	**0.39389270**	**0.00018126**	**0.00015134**
	DLF	BE	5.67643771	6.00891642	3.75504115	0.66938715	0.25998954
		PR	**0.00161349**	**0.00459807**	**0.02876226**	**0.00040468**	**0.00224394**
JP	SELF	BE	5.67214648	5.96637706	3.55392341	0.66873556	0.25937293
		PR	**0.05223551**	**0.16454251**	**0.37997292**	**0.00018111**	**0.00015129**
	DLF	BE	5.68135560	5.99395536	3.66083987	0.66900699	0.25995622
		PR	**0.00162094**	**0.00460102**	**0.02920545**	**0.00040572**	**0.00224380**
IP	SELF	BE	5.66010563	5.90345543	3.47706171	0.66738367	0.25985150
		PR	**0.05160368**	**0.15980341**	**0.35054842**	**0.00018099**	**0.00015105**
	DLF	BE	5.66922272	5.93052490	3.57787914	0.66765487	0.26043282
		PR	**0.00160817**	**0.00456443**	**0.02817799**	**0.00040520**	**0.00223210**

and their posterior risks assuming the UP, JP and the IP under SELF and DLF are shown in Table 5.

From Table 5, it is observed that the results obtained through real life data are compatible with simulation results. Table 5 also reveals that the performance of the IP is better than the UP and JP. Moreover, the results are relatively more precise under UP (JP) than JP (UP) with DLF (SELF). Also, it is observed that SELF (DLF) performance is better than DLF (SELF) for estimating proportion (component) parameters.

CONCLUSION

In this study, we have proposed a 3-component mixture of Pareto distributions to study a lifetime model. We have considered the Bayesian estimation of the 3-component mixture of Pareto distributions using the uninformative (uniform and Jeffreys') and informative (gamma) priors under SELF and DLF. We conducted a comprehensive simulation and a real life study to judge the relative performance of the Bayes estimators and also to deal with the problems of selecting the priors and loss functions at varying sample sizes and test termination times. The numerical results revealed that an increase in sample size or test termination time provides improved (in terms of closeness) and reliable (in terms of posterior risk) Bayes estimators. The extent of over-estimation (under-estimation) of the Bayes estimators of parameters is relatively smaller (larger) with relatively larger (smaller) test termination times (sample sizes) at different sample sizes (test termination times). Also, the extent of over-estimation (under-estimation) of the Bayes estimators of parameters is less for relatively larger values of component parameters. The posterior risks of Bayes estimators of component (proportion) parameters are smaller (larger) for smaller component parametric values under SELF (SELF and DLF). Moreover, as the sample size (test termination time) decreases (increase) the posterior risks of Bayes estimators of parameters increase (decrease) for a fixed test termination time (sample size). Furthermore, the DLF (SELF) is observed as a preferable choice for estimating the component (proportion) parameters. Finally, we conclude that the IP is more suitable and efficient prior under SELF for estimating the mixing proportion parameters. In case when DLF is selected, the IP is the preferable and efficient prior for estimating the component parameters. Furthermore, the same pattern is observed for real life data.

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APPENDIX

Table I: Bayes estimate (BE) and posterior risk (PR) using the UP with $\lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 5,\ p_1 = 0.5,\ p_2 = 0.3$

t	n	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
1.5	20	SELF	BE	5.293780	7.167190	10.34660	0.445858	0.316351
			PR	**8.796830**	**21.89270**	**64.78280**	**0.014928**	**0.013158**
		DLF	BE	6.674270	9.660760	14.56280	0.466265	0.358581
			PR	**0.203675**	**0.248921**	**0.289539**	**0.074871**	**0.116985**
	30	SELF	BE	4.649810	6.223970	7.984960	0.452320	0.314351
			PR	**4.683830**	**12.76210**	**25.71840**	**0.011412**	**0.009896**
		DLF	BE	5.695270	7.513440	10.39980	0.470642	0.347013
			PR	**0.163978**	**0.214910**	**0.251230**	**0.056930**	**0.094879**
	50	SELF	BE	4.158820	5.239980	6.544840	0.459023	0.313546
			PR	**2.516370**	**5.525710**	**10.38160**	**0.008028**	**0.006968**
		DLF	BE	4.845450	6.037990	7.804990	0.472357	0.337914
			PR	**0.116326**	**0.158489**	**0.193061**	**0.038453**	**0.065339**
	100	SELF	BE	3.712280	4.596760	5.607570	0.469456	0.311370
			PR	**1.110690**	**2.500540**	**4.549870**	**0.004774**	**0.004055**
		DLF	BE	3.975200	5.183670	6.502200	0.482546	0.322384
			PR	**0.069804**	**0.104267**	**0.128320**	**0.021626**	**0.039741**
	200	SELF	BE	3.409330	4.283070	5.256280	0.479015	0.307861
			PR	**0.504418**	**1.260720**	**2.263810**	**0.002761**	**0.002303**
		DLF	BE	3.619190	4.554520	5.687110	0.483589	0.316210
			PR	**0.038885**	**0.063263**	**0.079100**	**0.011980**	**0.023129**
2.0	20	SELF	BE	4.326770	5.877100	8.661690	0.462927	0.310159
			PR	**3.025480**	**8.994630**	**28.26120**	**0.011292**	**0.009699**
		DLF	BE	4.942350	7.109110	11.01430	0.486479	0.342898
			PR	**0.127336**	**0.175831**	**0.216095**	**0.051416**	**0.092724**
	30	SELF	BE	3.905800	5.234440	7.122760	0.471019	0.308955
			PR	**1.714070**	**4.684800**	**12.00940**	**0.008170**	**0.006987**
		DLF	BE	4.321150	6.102730	8.278430	0.488191	0.331122
			PR	**0.095580**	**0.136628**	**0.171694**	**0.036174**	**0.068835**
	50	SELF	BE	3.614970	4.712310	6.040530	0.478096	0.306863
			PR	**0.930862**	**2.429000**	**5.280310**	**0.005255**	**0.004462**
		DLF	BE	3.857660	5.202420	6.746530	0.489054	0.322045
			PR	**0.063631**	**0.096330**	**0.124057**	**0.022820**	**0.045472**
	100	SELF	BE	3.358970	4.278310	5.413960	0.486223	0.305294
			PR	**0.418080**	**1.109120**	**2.276240**	**0.002809**	**0.002375**
		DLF	BE	3.429900	4.498620	5.961290	0.491982	0.313282
			PR	**0.034472**	**0.056787**	**0.072859**	**0.011880**	**0.024854**
	200	SELF	BE	3.174230	4.165910	5.247270	0.492723	0.303053
			PR	**0.183232**	**0.562381**	**1.114090**	**0.001450**	**0.001219**
		DLF	BE	3.229560	4.301850	5.418120	0.494960	0.307223
			PR	**0.017354**	**0.031240**	**0.039848**	**0.005968**	**0.013045**

Table II: Bayes estimate (BE) and posterior risk (PR) using the JP with $\lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 5,\ p_1 = 0.5,\ p_2 = 0.3$

t	n	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
1.5	20	SELF	BE	4.776170	5.761560	7.783900	0.431126	0.318210
			PR	**7.688690**	**20.84890**	**57.77901**	**0.014516**	**0.013082**
		DLF	BE	5.993150	7.779150	10.93840	0.461278	0.361586
			PR	**0.212629**	**0.278813**	**0.354008**	**0.075679**	**0.117738**
	30	SELF	BE	4.383510	5.092650	5.953890	0.439036	0.316836
			PR	**4.372380**	**8.776920**	**18.00890**	**0.011127**	**0.009888**
		DLF	BE	5.170340	6.621740	8.181580	0.465072	0.348673
			PR	**0.166752**	**0.224551**	**0.290967**	**0.057275**	**0.095622**
	50	SELF	BE	4.016540	4.681760	5.485100	0.451435	0.314762
			PR	**2.248490**	**4.678600**	**8.667130**	**0.007740**	**0.006805**
		DLF	BE	4.518910	5.662230	6.963380	0.469097	0.335662
			PR	**0.116720**	**0.166958**	**0.214682**	**0.038475**	**0.065409**
	100	SELF	BE	3.656640	4.392330	5.188780	0.465229	0.311272
			PR	**1.052080**	**2.354530**	**4.287430**	**0.004681**	**0.003981**
		DLF	BE	3.971480	4.799080	5.972980	0.473696	0.325749
			PR	**0.070194**	**0.106783**	**0.137389**	**0.021968**	**0.039893**
	200	SELF	BE	3.432780	4.196060	4.916150	0.475185	0.308211
			PR	**0.503192**	**1.214890**	**2.149440**	**0.002698**	**0.002259**
		DLF	BE	3.577770	4.428510	5.429030	0.479554	0.317360
			PR	**0.039543**	**0.064669**	**0.083394**	**0.012289**	**0.023198**
2	20	SELF	BE	3.812270	4.794940	6.610910	0.458564	0.312636
			PR	**2.588380**	**6.533550**	**22.11650**	**0.011376**	**0.009875**
		DLF	BE	4.556390	6.280470	8.934860	0.480037	0.342670
			PR	**0.125459**	**0.171931**	**0.212535**	**0.051126**	**0.092032**
	30	SELF	BE	3.605970	4.691070	5.832640	0.466710	0.310229
			PR	**1.564680**	**4.411280**	**10.12920**	**0.008016**	**0.006741**
		DLF	BE	4.127120	5.411440	7.020090	0.483297	0.333727
			PR	**0.091608**	**0.130581**	**0.162581**	**0.035058**	**0.068736**
	50	SELF	BE	3.465300	4.428720	5.378130	0.476647	0.307542
			PR	**0.859103**	**2.257300**	**4.765290**	**0.005241**	**0.004456**
		DLF	BE	3.652560	4.777740	6.217610	0.487148	0.322075
			PR	**0.065808**	**0.102607**	**0.138358**	**0.023218**	**0.045777**
	100	SELF	BE	3.289030	4.137970	5.170360	0.485027	0.305456
			PR	**0.400117**	**1.067430**	**2.225230**	**0.002805**	**0.002369**
		DLF	BE	3.442520	4.324110	5.534790	0.489817	0.314314
			PR	**0.035066**	**0.058584**	**0.078099**	**0.011968**	**0.024879**
	200	SELF	BE	3.169930	4.076380	5.032030	0.490714	0.303835
			PR	**0.182605**	**0.543210**	**1.074960**	**0.001448**	**0.001218**
		DLF	BE	3.247290	4.163710	5.179290	0.492240	0.308836
			PR	**0.018145**	**0.032391**	**0.042490**	**0.006109**	**0.013102**

Table III: Bayes estimate (BE) and posterior risk (PR) using the IP with $\lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 5,\ p_1 = 0.5,\ p_2 = 0.3$

t	n	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
1.5	20	SELF	BE	3.540390	3.145740	3.509140	0.419505	0.329781
			PR	**1.698390**	**2.598450**	**4.739040**	**0.011098**	**0.010189**
		DLF	BE	3.851860	4.267690	4.695840	0.444939	0.362897
			PR	**0.151300**	**0.209193**	**0.274784**	**0.062250**	**0.087218**
	30	SELF	BE	3.502060	3.348100	3.638220	0.428640	0.325288
			PR	**1.484760**	**2.406050**	**4.098170**	**0.008860**	**0.008037**
		DLF	BE	3.796580	4.214960	4.771250	0.447001	0.354636
			PR	**0.120605**	**0.170223**	**0.230427**	**0.047807**	**0.070266**
	50	SELF	BE	3.462750	3.610000	4.023740	0.438785	0.323875
			PR	**1.154150**	**2.070960**	**3.987630**	**0.006345**	**0.005704**
		DLF	BE	3.591120	3.843450	4.377050	0.447387	0.331848
			PR	**0.317707**	**0.514406**	**0.805993**	**0.014587**	**0.017600**
	100	SELF	BE	3.774420	4.141330	4.811870	0.454378	0.340184
			PR	**0.087379**	**0.130739**	**0.176522**	**0.032737**	**0.052869**
		DLF	BE	3.442190	3.755000	4.372990	0.453626	0.320071
			PR	**0.717669**	**1.390290**	**2.681320**	**0.004011**	**0.003538**
	200	SELF	BE	3.549760	3.927520	4.662480	0.460277	0.322659
			PR	**0.199634**	**0.353660**	**0.567219**	**0.008835**	**0.011029**
		DLF	BE	3.671430	4.098480	4.862040	0.462532	0.330014
			PR	**0.055560**	**0.086941**	**0.119834**	**0.019346**	**0.033604**
2	20	SELF	BE	3.290520	3.273910	3.766050	0.440505	0.322807
			PR	**1.213430**	**2.245130**	**4.394200**	**0.009568**	**0.008512**
		DLF	BE	3.499770	3.895050	4.779170	0.460386	0.350403
			PR	**0.114067**	**0.166407**	**0.224762**	**0.047841**	**0.075442**
	30	SELF	BE	3.269420	3.498470	4.013880	0.451697	0.318647
			PR	**0.956750**	**1.884730**	**3.646740**	**0.007191**	**0.006319**
		DLF	BE	3.469220	3.925610	4.817520	0.467901	0.338580
			PR	**0.085790**	**0.130474**	**0.177488**	**0.034529**	**0.059074**
	50	SELF	BE	3.202550	3.654430	4.324700	0.465147	0.313187
			PR	**0.633400**	**1.377050**	**2.777700**	**0.004840**	**0.004178**
		DLF	BE	3.322200	3.853500	4.603320	0.468160	0.321798
			PR	**0.194414**	**0.355218**	**0.588500**	**0.010440**	**0.013184**
	100	SELF	BE	3.411920	4.051250	4.856560	0.473436	0.328112
			PR	**0.058334**	**0.091530**	**0.123583**	**0.022300**	**0.040763**
		DLF	BE	3.187560	3.769470	4.582490	0.477156	0.309742
			PR	**0.348156**	**0.836414**	**1.710630**	**0.002685**	**0.002292**
	200	SELF	BE	3.264830	3.928450	4.705900	0.479435	0.313647
			PR	**0.106237**	**0.211515**	**0.349885**	**0.005598**	**0.007322**
		DLF	BE	3.314320	4.028680	4.960830	0.482183	0.317820
			PR	**0.032333**	**0.053974**	**0.072611**	**0.011696**	**0.023185**

Table IV: Bayes estimate (BE) and posterior risk (PR) using the UP with $\lambda_1 = 5,\ \lambda_2 = 6,\ \lambda_3 = 7,\ p_1 = 0.5,\ p_2 = 0.3$

t	n	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
1.5	20	SELF	BE	7.124420	9.393800	13.03870	0.462682	0.309067
			PR	**9.070430**	**24.06560**	**78.00530**	**0.011687**	**0.010081**
		DLF	BE	8.171810	11.19340	64.60260	0.487594	0.341695
			PR	**0.134565**	**0.186591**	**0.231898**	**0.052884**	**0.096386**
	30	SELF	BE	6.510540	8.047660	10.44740	0.471592	0.306993
			PR	**5.040090**	**11.94470**	**30.84310**	**0.008412**	**0.007184**
		DLF	BE	7.145780	9.502000	12.33330	0.488852	0.330643
			PR	**0.101316**	**0.147375**	**0.187004**	**0.037374**	**0.071811**
	50	SELF	BE	6.061930	7.207320	8.812010	0.478147	0.306442
			PR	**2.815860**	**6.274300**	**12.69820**	**0.005485**	**0.004675**
		DLF	BE	6.511030	8.098060	10.07360	0.489230	0.321866
			PR	**0.067828**	**0.103964**	**0.137272**	**0.023751**	**0.047472**
	100	SELF	BE	5.543270	6.686620	7.761700	0.486754	0.303927
			PR	**1.243740**	**2.991410**	**5.479060**	**0.002965**	**0.002492**
		DLF	BE	5.781310	7.071740	8.418490	0.492033	0.312453
			PR	**0.037282**	**0.061769**	**0.083897**	**0.012474**	**0.026219**
	200	SELF	BE	5.339600	6.278260	7.334320	0.490692	0.303415
			PR	**0.597448**	**1.450400**	**2.658400**	**0.001556**	**0.001306**
		DLF	BE	5.409390	6.604880	7.737650	0.495218	0.306814
			PR	**0.019786**	**0.035288**	**0.048261**	**0.006402**	**0.013976**
2	20	SELF	BE	6.407360	8.427200	11.78010	0.474120	0.305951
			PR	**4.798620**	**14.10020**	**43.86500**	**0.010555**	**0.008995**
		DLF	BE	7.208350	9.805270	14.21730	0.495131	0.335764
			PR	**0.095387**	**0.140164**	**0.181673**	**0.045056**	**0.087649**
	30	SELF	BE	5.986810	7.657960	9.746290	0.481403	0.304202
			PR	**2.816670**	**7.745230**	**18.27940**	**0.007472**	**0.006334**
		DLF	BE	6.405000	8.433060	11.20070	0.496097	0.325599
			PR	**0.068350**	**0.104585**	**0.139633**	**0.031432**	**0.064026**
	50	SELF	BE	5.583860	6.953920	8.480530	0.486734	0.303497
			PR	**1.514350**	**3.867450**	**8.387800**	**0.004731**	**0.004000**
		DLF	BE	5.898570	7.372440	9.436400	0.496687	0.316615
			PR	**0.044319**	**0.070327**	**0.097111**	**0.019576**	**0.041637**
	100	SELF	BE	5.328560	6.404100	7.667730	0.492696	0.302070
			PR	**0.696689**	**1.700300**	**3.472690**	**0.002463**	**0.002076**
		DLF	BE	5.440250	6.686570	8.269020	0.498027	0.308928
			PR	**0.023371**	**0.039023**	**0.054250**	**0.010040**	**0.022257**
	200	SELF	BE	5.193900	6.204240	7.279080	0.496061	0.301209
			PR	**0.329232**	**0.816373**	**1.593910**	**0.001257**	**0.001057**
		DLF	BE	5.219720	6.348450	7.526290	0.498650	0.304666
			PR	**0.011864**	**0.020553**	**0.028884**	**0.005080**	**0.011518**

Table V: Bayes estimate (BE) and posterior risk (PR) using the JP with $\lambda_1 = 5,\ \lambda_2 = 6,\ \lambda_3 = 7,\ p_1 = 0.5,\ p_2 = 0.3$

t	n	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
1.5	20	SELF	BE	6.547750	7.713170	9.842990	0.459126	0.310361
			PR	**7.617660**	**19.32880**	**75.13260**	**0.011571**	**0.010033**
		DLF	BE	7.453140	9.899450	13.08370	0.483967	0.342802
			PR	**0.143596**	**0.211844**	**0.281848**	**0.053770**	**0.096209**
	30	SELF	BE	6.160160	6.933450	8.323450	0.467271	0.308640
			PR	**4.713350**	**10.06960**	**22.54870**	**0.008410**	**0.007164**
		DLF	BE	6.884290	8.439090	10.34570	0.484572	0.331435
			PR	**0.104031**	**0.160101**	**0.214513**	**0.037757**	**0.071587**
	50	SELF	BE	5.830590	6.675700	7.658750	0.475033	0.306941
			PR	**2.625140**	**5.725880**	**11.20090**	**0.005451**	**0.004670**
		DLF	BE	6.266110	7.435330	8.838270	0.486721	0.321738
			PR	**0.069136**	**0.110086**	**0.153820**	**0.024038**	**0.047685**
	100	SELF	BE	5.502640	6.421160	7.220050	0.483955	0.304861
			PR	**1.217280**	**2.814820**	**5.102490**	**0.002941**	**0.002480**
		DLF	BE	5.686510	6.781550	8.102670	0.490581	0.312485
			PR	**0.037664**	**0.063760**	**0.088594**	**0.012509**	**0.026222**
	200	SELF	BE	5.280110	6.192940	7.152950	0.491298	0.302151
			PR	**0.571080**	**1.410380**	**2.592280**	**0.001542**	**0.001289**
		DLF	BE	5.417380	6.391220	7.407870	0.493021	0.307733
			PR	**0.019992**	**0.035579**	**0.050234**	**0.006464**	**0.013986**
2	20	SELF	BE	5.904780	7.286010	8.924810	0.473698	0.305431
			PR	**4.435480**	**12.34230**	**31.84470**	**0.010540**	**0.008979**
		DLF	BE	6.390760	8.813630	11.40400	0.496450	0.333932
			PR	**0.103591**	**0.161855**	**0.221313**	**0.045071**	**0.088585**
	30	SELF	BE	5.607610	6.747970	8.273390	0.479934	0.305196
			PR	**2.650110**	**6.693960**	**15.89740**	**0.007423**	**0.006318**
		DLF	BE	6.018410	7.480280	9.946920	0.496428	0.324981
			PR	**0.071944**	**0.115268**	**0.161215**	**0.031388**	**0.064296**
	50	SELF	BE	5.443970	6.387330	7.443980	0.486847	0.303122
			PR	**1.462580**	**3.461830**	**7.182980**	**0.004727**	**0.003995**
		DLF	BE	5.598420	6.934720	8.492240	0.496631	0.316037
			PR	**0.045877**	**0.074762**	**0.107125**	**0.019610**	**0.041798**
	100	SELF	BE	5.231700	6.157540	7.339550	0.492535	0.302223
			PR	**0.677330**	**1.607840**	**3.346260**	**0.002461**	**0.002075**
		DLF	BE	5.372230	6.452630	7.681150	0.497597	0.308789
			PR	**0.023828**	**0.040299**	**0.057718**	**0.010062**	**0.022279**
	200	SELF	BE	5.135460	6.135900	7.065860	0.496024	0.301026
			PR	**0.323280**	**0.806709**	**1.547360**	**0.001256**	**0.001056**
		DLF	BE	5.171470	6.281310	7.292690	0.498390	0.304741
			PR	**0.012093**	**0.021039**	**0.030217**	**0.005094**	**0.011525**

Table VI: Bayes estimate (BE) and posterior risk (PR) using the IP with $\lambda_1 = 5,\ \lambda_2 = 6,\ \lambda_3 = 7,\ p_1 = 0.5,\ p_2 = 0.3$

t	n	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
1.5	20	SELF	BE	4.344580	4.163800	4.510430	0.440226	0.323124
			PR	**2.364250**	**3.422250**	**6.026100**	**0.009637**	**0.008610**
		DLF	BE	4.803140	4.950240	5.694260	0.461889	0.348577
			PR	**0.112536**	**0.167487**	**0.224402**	**0.048485**	**0.077899**
	30	SELF	BE	4.629970	4.581100	5.001450	0.451793	0.318507
			PR	**1.979140**	**3.160450**	**5.651290**	**0.007317**	**0.006454**
		DLF	BE	5.045190	5.248880	6.010000	0.467706	0.338528
			PR	**0.083293**	**0.127728**	**0.177305**	**0.034874**	**0.059984**
	50	SELF	BE	4.847650	5.082750	5.550960	0.463670	0.313236
			PR	**1.469430**	**2.681770**	**4.717730**	**0.004928**	**0.004276**
		DLF	BE	5.144300	5.289950	5.917710	0.469229	0.320195
			PR	**0.295634**	**0.493826**	**0.787210**	**0.010610**	**0.013566**
	100	SELF	BE	5.232410	5.539640	6.323000	0.474140	0.327990
			PR	**0.056721**	**0.090586**	**0.129076**	**0.022542**	**0.041716**
		DLF	BE	5.148800	5.496970	6.035040	0.475709	0.309810
			PR	**0.920786**	**1.802610**	**3.238200**	**0.002752**	**0.002364**
	200	SELF	BE	5.214860	5.642280	6.295200	0.478188	0.313253
			PR	**0.173847**	**0.315682**	**0.507198**	**0.005809**	**0.007615**
		DLF	BE	5.353490	5.779430	6.614670	0.481350	0.317637
			PR	**0.032852**	**0.055248**	**0.079737**	**0.012072**	**0.024076**
2	20	SELF	BE	4.218210	4.345820	4.696350	0.453757	0.317675
			PR	**1.843500**	**3.165140**	**5.566630**	**0.009021**	**0.007899**
		DLF	BE	4.557570	5.092090	5.868930	0.473419	0.342212
			PR	**0.092122**	**0.140153**	**0.173028**	**0.042233**	**0.072911**
	30	SELF	BE	4.467630	4.733490	5.233320	0.464822	0.313208
			PR	**1.452700**	**2.695780**	**4.973130**	**0.006688**	**0.005786**
		DLF	BE	4.802670	5.297970	6.106920	0.479211	0.331872
			PR	**0.066256**	**0.104003**	**0.136759**	**0.030094**	**0.055766**
	50	SELF	BE	4.755990	5.180880	5.881180	0.475648	0.309511
			PR	**1.035030**	**2.050860**	**3.982750**	**0.004401**	**0.003770**
		DLF	BE	4.864600	5.336320	6.160650	0.480478	0.315480
			PR	**0.208903**	**0.372767**	**0.620162**	**0.009207**	**0.012067**
	100	SELF	BE	4.952820	5.576690	6.364670	0.485449	0.321170
			PR	**0.042793**	**0.069235**	**0.097033**	**0.019068**	**0.038004**
		DLF	BE	4.911040	5.548710	6.325220	0.486128	0.305606
			PR	**0.572759**	**1.244130**	**2.397830**	**0.002373**	**0.002015**
	200	SELF	BE	4.951610	5.685850	6.485940	0.488747	0.308535
			PR	**0.114149**	**0.219229**	**0.362184**	**0.004870**	**0.006559**
		DLF	BE	5.025300	5.823170	6.765560	0.491321	0.311822
			PR	**0.022859**	**0.038253**	**0.054116**	**0.009933**	**0.021144**