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RESEARCH ARTICLE

Ising model across planar lattices

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Abstract: The Ising model on a two dimensional lattice with pentagon tiling and greater number of next to neighbours interaction is equivalent to the Ising model on the centered square lattice with nearest and next to nearest neighbour non-crossing interactions. We obtain exact expressions on calculation of critical temperature with ferromagnetic and anti-ferromagnetic interactions. A direct anti-ferromagnetic bond of strength αL can give rise to the transition temperatures for appropriate α. We apply the star-triangle mapping transformation technique and decorating transform to Ising model.

Keywords: Exact results, Ising model, square lattice, transformations.

INTRODUCTION

Exact solution of the planar Ising model on a square lattice with nearest neighbour interactions were obtained in 1944[1]. Exact solutions for a number of more complex Lattices[2-5] and exact solution of the Ising model on the 4-6 lattice triangle, pentagonal and hexagonal lattices were obtained[6,7]. Some attention has been made to the model on hierarchical lattices[8]. It is known, in fact, that all planar Ising models with non-crossing bonds are, in principle, solvable. However, it is still of interest to search for other exactly solved models towards particular interactions and to compute the critical point for those particular lattices with ferromagnetic and anti-ferromagnetic interactions. For this, various transformation methods like duality, star-triangle and decoration-iteration to the Ising model on

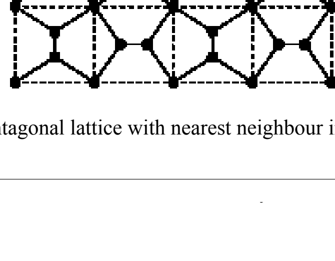

Figure 1: Pentagonal lattice with nearest neighbour interactions

regular two-dimensional lattices have been investigated in the past[9,10]. We consider a model with competing ferromagnetic and anti-ferromagnetic interactions acting on a two-dimensional lattice with pentagon tiling.

We consider an Ising model on a planar lattice where tiling is achieved by polygons, namely triangles, pentagons and septogons. The lattice view as a model by turning 90^0 in the neighbouring square plaquette.

Inner spins in every plaquette of the pentagon lattice are replaced with triangles of spins linked between themselves at one identical site and additionally connected with their remaining sites with a pair of neighbouring corners of the square lattice. We have a lattice that is exactly mapped onto a pentagonal lattice that is illustrated in Figure 2.

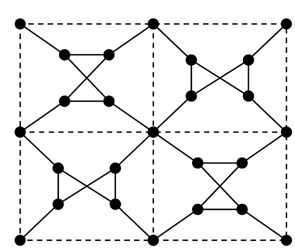

Figure 2: Star-triangle transformation for pentagonal lattice

Ordering the squares is in the same manner. Applying the transformation method to the lattice we have with triangles and septogones, we get the same situation.

There are two different interaction parameters J and J' coupling particular nearest neighbour pairs of spins. Let s_1, s_2 , s_3 and s_4 be corner spins located at the sites of a lattice and σ_1 and σ_2 be inner spins. The Ising Hamiltonian defined upon the underlying elementary plaquette is given by:

$$H = -J[\sigma_1(s_1 + s_2) + \sigma_2(s_3 + s_4)] - J^{/}\sigma_1\sigma_2$$

Additional spin s is introduced at each internal bond.

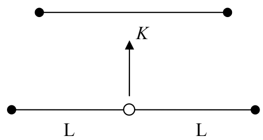

Figure 3: Effective coupling k

The basic decoration-iteration transform is shown in Figure 3, where summing over the central spin s with coupling L gives rise to a new effective coupling K between the primary vertex spins σ_1, σ_2 :

$$\sum_s \exp[L\, s(\sigma_1 + \sigma_2)] = A \exp(K\, \sigma_1\sigma_2) \quad \text{... (1)}$$

Here σ_1, σ_2 = ±1, $A = e^{-k^{/}/2}$. The effective coupling parameter $k^{/} = \beta J^{/}$, $\beta = \frac{1}{k_\beta T}$. K_β is the Boltman's Constant and T is the absolute temperature and we get $A^2 = 4\cosh(2L)$.

The effective interaction between the spins s and σ_1 , σ_2 is defined by

$$K = \frac{1}{2}\log[\cosh(2L)] \quad \text{.... (2)}$$

This is limit to the case $J^{/}/2\beta T > 0$.

Summation in the partition function over the internal spins σ_1, σ_2 is the stat-triangle transformation and effective interaction follows the expression

$$\sum_{\sigma_1\sigma_2} e^{L\sigma + k^{/}(s_1+s_2)\sigma_1} = 2\cosh[L\sigma + \beta J(s_1 + s_2)] = \beta e^{k_1(s_1+s_2)\sigma + k_2 s_1 s_2}$$

$k = \beta J$, $\sigma = \pm 1$, $s_1 = \pm 1$, $s_2 = \pm 1$

We get an Ising model on the centered square lattice with the nearest-neighbours interaction k_1 and non-crossing diagonal interaction k_2.

The isotropic nearest-neighbours interaction

$$k_1 = \frac{1}{4}\left[\frac{\cosh(L + 2k^{'})}{\cosh(L - 2k^{'})}\right]$$

and the non-crossing diagonal interaction between next to nearest-neighbours

$$k_2 = \frac{1}{4}\left[\frac{\cosh 2L + \cos 4k^{'}}{\cosh(2L + 1)}\right]$$

We introduce one decorating spin as well as the direct anti-ferromagnetic bond. If a direct anti-ferromagnetic bond is stirred into the mix as well for good measure this becomes

$$K = -\alpha L + \frac{1}{2}\log[\cosh(2L)] \quad \text{.... (3)}$$

In this case summing over the intermediate spins gives

$$K = -\alpha L + \frac{1}{2}\log\left[\frac{(\exp(2L)+1)^2 + (\exp(2L)-1)^2}{(\exp(2L)+1)^2 - (\exp(2L)-1)^2}\right] \quad \text{.... (4)}$$

along with an equation for the normalization factor

$$A^2 = 4\,\frac{[\sinh(2L)^2]}{\sinh(2K)} \quad \text{.... (5)}$$

Ising spins, which distribute on a plane decorated lattice and couple with their neighbouring spins by antiferromagnetic exchange interaction, are found to show a number of varieties in its magnetic properties[10]. Basically, the form of the transformation in both equations (3) and (4) meant that *three* transitions could occur for an Ising model.

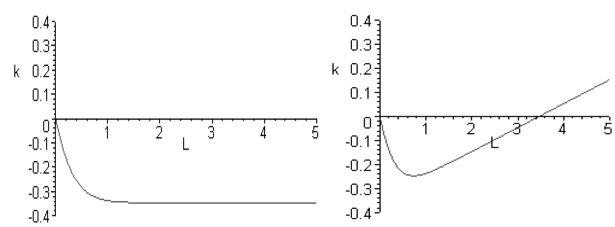

Figure 4: (a) k is ≈ -0.38 when α →1 Figure 4: (b) α increased

For the square lattice Ising model for an individually decorated model, the minimum value of K is ≈ -0.318 when $\alpha \to l$ as in Figure 4(a). As α increased further $K(L)$ eventually becomes monotonically increasing function illustrated in Figure 4(b) where ferromagnetic transition disappears.

DISCUSSION

In conclusion, the Ising model on a pentagonal lattice with nearest neighbour and next to nearest neighbour interactions are described by two parameters and is equivalent to the Ising model on the centered square lattice. Exact expressions were obtained to compute critical temperature when a model with competing ferromagnetic and anti-ferromagnetic interactions acting on a two-dimensional lattice with pentagon tiling. It is shown that lattice with a large number of nearest neighbour interaction parameters will also be exactly solvable.

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