

A MATHEMATICAL MODEL OF A PROBLEM ON LAND ALIENATION TO LANDLESS FARMERS

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Abstract: When new townships are created, for example, due to new irrigation facilities, and land is alienated equally among landless farmers, it becomes necessary for the planners to determine, before hand, how far apart the trunk roads are to be and what the dimensions should be of the rectangular plot of land alienated to each farmer, after predetermining the number of hectares each farmer is to be given and the breadth of each minor roadway serving the plots. In the following it is assumed that the land is flat, the trunk roads are such that one set is parallel to one another and the others are perpendicular to the former. This generates large rectangular blocks of land which are to be parcelled out among the farmers. One problem that is discussed here is the minimization of the average cost per farmer for fencing their plots in each large rectangular block. The other problem is that of minimizing the total cost of fencing all the plots, again in each large rectangular block. Both the above problems have been solved in a very general way and specific solutions have been worked out when

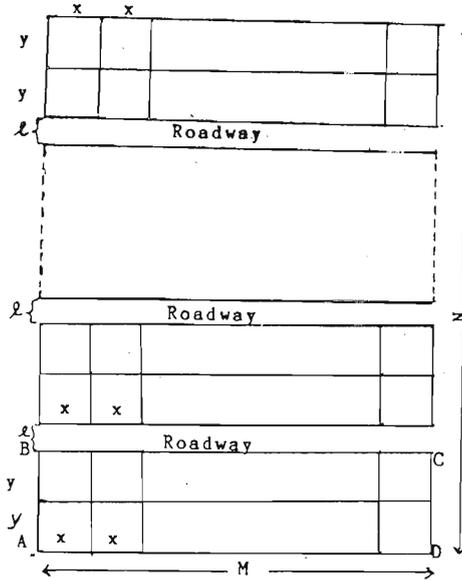
- (i) the trunk roads are about 1,500 meters apart;
- (ii) the minor roads serving the plots are 6 meters wide and
- (iii) each farmer is given 2 hectares of land.

Since the width of a minor road (6 meters) is very small in comparison to the distance between the trunk roads (about 1,500 meters) it is found that, although the two problems discussed are algebraically very different, their final solutions are very close to each other.

When new land is made available for agriculture through the development of irrigation facilities etc. and when government policy is to parcel out the land in small units, such as 1 or 2 hectares, among a large number of landless families, the demarcation of the boundaries of the blocks to be so allotted raises some interesting but simple mathematical problems. Government lays down its policy regarding trunk roads. The obvious plan is to have one set of such roads, all parallel, say running north to south at a suitable distance M meters apart, and another set of roads perpendicular to the first set, at a separation of N meters apart where M and N are determined by the government as a matter of policy. They may be the same and may be, for example, about 1,500 meters. This process of demarcating the trunk roads creates large rectangular blocks of land with sides M meters and N meters, and these are the blocks which are to be parcelled out to farmer families at A square meters per family. If one hectare is allotted per family, then

$$A = 100^2 \text{ sq. meters.}$$

The government may decide to block out such land parcels in the following manner. Assuming the land to be generally flat as is the land in the newly irrigated regions of Sri Lanka, the blocks must be rectangular and of the same dimensions.



The side of length M is divided into m equal parts each of length x meters. n blocks of land such as $ABCD$ are created together with $n-1$ minor roadways each of breadth l meters, say of the order of 6 meters. Each such block is further broken down into $2m$ rectangular blocks with sides x meters, y meters and each farmer family will be given one such block. The constraints imposed by the government as a matter of policy give rise to the following equations.

$$\left\{ \begin{array}{l} xy = A \\ mx = M \\ n \cdot 2y + (n-1)l = N \end{array} \right\} \longrightarrow (1)$$

In the above equations, A, M, N, l are determined as matters of policy. m, n have to be positive integers.

Taking m to be an arbitrary positive integer, x is determined by the second equation of (1). Then y is determined by the first equation of (1). But, the third equation may not now yield a positive integral value for n .

We will discuss the resolution of such problems later, but consider two optimization problems which can arise here.

Problem 1: The government decides to fence all the blocks before giving them to the farmers and wishes to find out the optimal dimensions x, y which will minimize the total length of fencing, which is $3m^2x + 2y(m + 1)n$, subject to the conditions of equations (1).

Problem 2: The government decides to fence all the blocks before giving them to the farmers but wants each farmer to share the cost equally and to minimize this share. This is the problem of minimizing $[3m^2x + 2y(m + 1)n] / 2mn$ subject to the conditions of equations (1).

Since Problem 2 is much easier to handle, we will consider it first.

The function F that has to be minimized can be expressed solely in terms of m because

$$\begin{aligned} x &= M/m \text{ and } y = A/x = Am/M \\ \therefore F &= 3M/2m + (m + 1)A/M \\ &= A/M + Am/M + 3M/2m \\ &\geq A/M + 2\sqrt{3A/2}, \end{aligned} \quad \text{the equality}$$

occurring when $m = \sqrt{3M^2/2A}$.

The minimum cost per farmer will be $A/N + 2\sqrt{3A/2}$ if the minor roadways of breadth l are constructed parallel to the side of length N . Thus, from the point of view of minimizing the cost per farmer, it is better to lay those minor roadways parallel to the longer side.

Thus, the cost per farmer will be minimized when

$$m = M \sqrt{3/2A} \quad \text{—————} \quad (2)$$

$$x = \sqrt{2A/3} \quad \text{—————} \quad (3)$$

$$y = \sqrt{3A/2} \quad \text{—————} \quad (4)$$

$$n = [N + l] / [2\sqrt{3A/2} + l] \quad \text{—————} \quad (5)$$

The area R_1 occupied by the minor roadways is $lM(n-1)$ which is equal to $lM[(N + l) / (2\sqrt{3A/2} + l) - 1]$.

If the minor roadways are laid parallel to the side of length N , the area R_2 occupied by the minor roadways is

$$R_2 = lN \left[\frac{M + l}{2\sqrt{3A/2} + l} - 1 \right]$$

$$\begin{aligned}
 R_1 - R_2 &= \frac{l}{2\sqrt{3A/2} + l} [M(N + l) - N(M + l)] - l(M - N) \\
 &= \frac{l^2(M - N)}{2\sqrt{3A/2} + l} - l(M - N) \\
 &= \frac{l(M - N)}{2\sqrt{3A/2} + l} \{l - 2\sqrt{3A/2} - l\} \\
 &= \frac{2\sqrt{3A/2} - l(N - M)}{2\sqrt{3A/2} + l}
 \end{aligned}$$

Thus if $N > M$, not only will the cost per family be further reduced by laying the minor roads parallel to the longer side of length N , but it also has the additional advantage of allocating less land for the minor roadways. Since agricultural productivity is the most important factor, it seems more desirable, in more than one way, to lay the minor roads parallel to the longer side, so that more families will be able to receive land, A being a constant.

It is most unlikely that the above values of m and n would turn out to be positive integers. If it happens to be so, we have a beautiful but simple solution to the problem. If not, what are we going to do?

M, N are generally fixed. It should therefore be possible to adjust l or A or both so that the resulting m, n would be whole numbers. In fact, by the minimum possible upward adjustment of A , which will be beneficial to the individual farmer, it can be ensured that m becomes a positive integer. Thereafter l can be adjusted upwards or downwards so that n becomes a positive integer and l is not less than the minimum value it should take so that the minor roads of breadth l will not cease to serve their purposes.

Another possibility is to adjust M and N at the planning stage so that m, n would come out as positive whole numbers.

This latter solution is better because the breadth of each minor road remains unchanged and can be kept at its optimal breadth from the utility point of view and therefore, the maximum possible amount of land is available for alienation.

It is interesting to note that x, y are dependent only on A .

As a typical case, let us take $M = N = 1500$ meters, $l = 6$ meters and $A = 2$ hectares. i.e $A = 2 \times 100^2$ meters².

$$\begin{aligned}
 \text{Then } m &= 12.99 \text{ and} \\
 n &= 4.27
 \end{aligned}$$

Now we can adjust m to 13 and n to 4 say. Then $M = 1501.11m$ and $N = 1403.64m$. In this solution, the minor roadways are also laid parallel to the longer side. If we adjust m to 13 and n to 5, more peasants will be accommodated in this block and then $M = 1501.11 m$ and $N = 1756.05 m$, and the minor roadways are laid parallel to the shorter side. This solution is therefore less preferable to the former.

In either case $x = 115.47$ meters and $y = 173.21$ meters. The above two solutions give rise to the following useful data.

| m | n | M meters | N meters | Total fence length meters | Fence length/ farmer meters |
|-----|-----|---------------|---------------|---------------------------------|-----------------------------------|
| 13 | 4 | 1501.11 | 1403.64 | 37412.84 | 359.74 |
| 13 | 5 | 1501.11 | 1756.05 | 46766.05 | 359.74 |

It is not accidental that the fence length/farmer is the same in both cases because it is, in fact, $A/M + 2\sqrt{3A/2}$.

Let us now discuss the other problem where the total cost of fencing is borne by the government and this is to be minimized. The function G that has to be minimized can be expressed in terms of y because $x = A/y$, $mx = M$, $n = (N + l) / (2y + l)$ and $m = My/A$.

$$\begin{aligned} \therefore G(y) &= 3M \frac{N+l}{2y+l} + 2y \left(\frac{My}{A} + 1 \right) \frac{N+l}{2y+l} \\ &= \frac{N+l}{2y+l} \left\{ 3M + \frac{2M}{A} y^2 + 2y \right\} \end{aligned}$$

Differential calculus now be used to determine the maxima and minima of this function but a much more elementary method gives results more easily.

Taking $2y + l = z$,

$$\begin{aligned} G(y) = H(z) &= \frac{N+l}{z} \left\{ 3M + \frac{2M}{A} \frac{z^2 - 2zl + l^2}{4} + z - l \right\} \\ &= (N+l) \left\{ \left(1 - \frac{Ml}{A} \right) + \frac{Mz}{2A} + \frac{3M + \frac{Ml^2}{2A} - l}{z} \right\} \\ &\geq (N+l) \left\{ \left(1 - \frac{Ml}{A} \right) + 2\sqrt{\frac{M}{2A} \left(3M + \frac{Ml^2}{2A} - l \right)} \right\} \end{aligned}$$

provided that $3M + \frac{Ml^2}{2A} - l \geq 0$

The equality occurs here when

$$z = \sqrt{\frac{2A}{M} \left(3M + \frac{Ml^2}{2A} - l\right)} \quad \text{-----} \quad (6)$$

$$\begin{aligned} \text{Now, } 3M + \frac{Ml^2}{2A} - l &= \frac{M}{2A} \left(l^2 - \frac{2Al}{M} + 6A\right) \\ &= \frac{M}{2A} \left\{ \left(l - \frac{A}{M}\right)^2 + \frac{A}{M^2} (6M^2 - A) \right\} \end{aligned}$$

and in any realistic situation A will be very much less than $6M^2$

$$\text{Thus } G(y) \text{ has } (N+l) \left\{ \left(1 - \frac{Ml}{A}\right) + 2\sqrt{\frac{M}{2A} \left(3M - l + \frac{Ml^2}{2A}\right)} \right\}$$

as its least value when

$$y = \frac{1}{2} \sqrt{6A - \frac{2Al}{M} + l^2} - \frac{l}{2} \quad \text{-----} \quad (7)$$

This is also a positive value because in any realistic situation l will be very much smaller than $3M$ and,

$$6A - \frac{2Al}{M} + l^2 = l^2 + \frac{2A}{M} (3M - l) > l^2$$

$$\text{Then } x = \frac{A}{y} = 2A / \left\{ \sqrt{6A - \frac{2Al}{M} + l^2} - l \right\}$$

$$= \frac{2A \left\{ \sqrt{6A - \frac{2Al}{M} + l^2} + l \right\}}{2A (3M - l)/M}$$

$$= \frac{M \left\{ \sqrt{6A - \frac{2Al}{M} + l^2} + l \right\}}{3M - l} \quad \text{-----} \quad (8)$$

$$m = \frac{M}{2A} \left\{ \sqrt{\left(6A - \frac{2Al}{M} + l^2\right)} - l \right\} \longrightarrow (9)$$

$$\text{and } n = (N + l) / \sqrt{6A - \frac{2Al}{M} + l^2} \longrightarrow (10)$$

Since, in a realistic situation l is very much smaller than M , the above equations (7), (8), and (9) and (10) can be approximated to

$$y_0 = \sqrt{3A/2},$$

$$x_0 = \sqrt{2A/3},$$

$$m_0 = M \sqrt{3/2A} \text{ and}$$

$$n_0 = N/\sqrt{6A},$$

where, in the last approximation, it has also been assumed that l is very much less than N . This n_0 is exactly what one would get if equation (5) for n is subjected to the same approximation. The above values are, therefore, the same as the corresponding values in the solution to the other problem where the farmers share the cost of fencing equally.

As in the previous case, the above values for m, n are very unlikely to be positive integers. Here too, we can overcome the problem by one of the two ways we adopted in the previous case.

Let us apply our results to the specific case considered earlier where $M = N = 1500$ meters, $l = 6$ meters and $A = 2$ hectares = 2×100^2 meters.²

This gives $m = 12.76$

$$n = 4.35$$

$$x = 117.57 \text{ meters and}$$

$$y = 170.12 \text{ meters.}$$

We may now follow the same procedure as we adopted earlier and adjust m to 13 and n to 4 and determine what M and N should be and incorporate this result at the stage at which the major trunk roads are planned.

Equation (9) reduces to a quadratic equation in M as is seen by the following argument. Equation (9) gives

$$\begin{aligned} 2Am/M + l &= \sqrt{6A - 2Al/M + l^2} \\ \therefore 4A^2 m^2/M^2 + 4Am l/M &= 6A - 2Al/M \\ \therefore 6M^2 - 2l(1 + 2m)M - 4Am^2 &= 0 \\ \therefore 3M^2 - l(1 + 2m)M - 2Am^2 &= 0. \end{aligned}$$

This equation has one and only one positive real root for M which is

$$M = \left\{ l(1 + 2m) + \sqrt{l^2(1 + 2m)^2 + 24Am^2} \right\} / 6 \quad \text{—————} \quad (11)$$

Also from (9) and (10),

$$\begin{aligned} (2Am/M) + l &= (N + l)/n \\ \therefore N &= (2Amn/M) + (n - 1)l \quad \text{—————} \quad (12) \end{aligned}$$

Equation (11) determines M and thereafter, equation (12) determines N . The values of x and y have to be recalculated using (7) and (8). This will become easier if we note that

$$y = \frac{1}{2} \left[\frac{N + l}{n} - l \right] = \frac{Am}{M} \quad \text{—————} \quad (13)$$

$$x = \frac{M}{3M - l} \left[\frac{N + l}{n} + l \right] \quad \text{—————} \quad (14)$$

Using the values $m = 13$, $n = 4$, $A = 2 \times 100^2$ meters², equation (11) gives $M = 1528.35$ meters. Then equation (12) gives $N = 1378.94$ meters. Also $x = 117.57$ and $y = 170.12$.

As we anticipated earlier, the final solutions to the two problems are not too different arithmetically although they are algebraically different, the reasons being the relative smallness of l in relation to M and N .

One last thing that we can do is to compare the total fence length in the two cases. This is $3xm + 2y(m + 1)n$ which came to 37,412.84 meters when the families were sharing the total cost equally, and m, n were 13 and 4 respectively. In the present case it is 37,394.36 which is not significantly different from the former as would have been anticipated. The difference is because the block size in the two cases, i.e. the values of MN in the two cases are slightly different.