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ON ORDINARY LIMITABILITY FACTORS FOR CESARO MEANS

C. YOGACHANDRAN
Department of Mathematics and Statistics, University of Maiduguri, Nigeria.

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Abstract : This paper deals with the problem of finding necessary and sufficient conditions in order that, for some l', $f(x)g(x) \sim l'x^{p+q}$ (C, μ) whenever $f(x) \sim l\,x^p$ (C, λ) for some l, where $\mu \geqslant \lambda \geqslant 1$, $p > -1$, $p + q > -1$ and $f \in L_{loc}$. This problem is a generalization of a problem considered earlier,* in which μ, λ were replaced by positive integers r, k, $r \geqslant k$ and l and l' were zero.

1. Introduction

Let f be a real function with domain $\leqslant [1, \infty)$. If $f \in L_{loc}$ (i.e. f is locally Lebesgue integrable) and $\lambda > 0$, we define

$$f_\lambda(x) = I_\lambda f(x) = \frac{1}{\Gamma(\lambda)}\int_1^x (x-t)^{\lambda-1} f(t)dt, \text{ and set } f_o(x) = f(x).$$

we say that f(x) is Cesaro limitable of order λ in the ordinary sense to l, written $f(x) \to l(C, \lambda)$ if $\Gamma(\lambda + 1)x^{-\lambda}f_\lambda(x) \to l$ as $x \to \infty$. More generally, if $p > -1$. $l \neq 0$, we write $f(x) \sim lx^p$ (C, λ) if

$$\frac{\Gamma(\lambda+p+1)}{\Gamma(P+1)} x^{-p-\lambda} f_\lambda(x) \to l \text{ as } x \to \infty.$$

If p is real, we write $f(x) = 0(x^p)(C, \lambda)$[or $o(x^p)(C, \lambda)$] if $x^{-p-\lambda}f_\lambda(x) = 0(1)$ [or $o(1)$] as $x \to \infty$.

In (5), we found conditions necessary and sufficient in order that $f(x)g(x) = o(x^{p+q})(C, r)$ whenever $f(x) = o(x^p)(C, k)$, where r, k are non-negative integers and $r \geqslant k$. (See (5), Theorem 1). This is the integral analogue of a theorem given in (3). The following theorem is a direct consequence of the above Theorem 1.

THEOREM A : Let $r, k \in /N_v$, $r \geqslant k$, $p > -1$, $p+q > -1$. Also let $f \in L_{loc}$ and $g \in L^\infty_{loc}$ if $k = 0$, $r \geqslant 1$, and $g^{k-1} \in AC_{loc}$ if $\equiv k \geqslant 1$.

Then, conditions necessary and sufficient in order that for some l', $f(x)g(x) \sim l'x^{p+q}(C, r)$ whenever $f(x) \sim lx^p(C, k)$ for some l are :

* (5), Theorem 1.

(a) $(i)_a$ $\int_1^x |\phi(t)|\, dt = 0(x)$ as $x \to \infty$,

$(ii)_a$ $\phi(x) \to l''$ as $x \to \infty$ (C, r) for some l'', where $\phi(x) = x^{-q} g(x)$, in the case $k = 0$, $r \geqslant 1$ and

(b) $(i)_b$ $\phi(x) = 0(1)$ as $x \to \infty$,

$(ii)_b$ $\int_1^x t^{k-q} |g^k(t)|\, dt = 0(x)$ as $x \to \infty$,

$(iii)_b$ $\phi(x) \to l''$ (C, r) as $x \to \infty$,

in the case $r \geqslant k \geqslant 1$.

In this paper, we generalise Theorem A. We replace the integers r, k by real numbers μ, λ respectively, where $\mu \geqslant \lambda \geqslant 1$. We also drop the restriction $g^{k-1} \in AC_{loc}$. We prove the following theorem.

THEOREM B : Let $\mu \geqslant \lambda \geqslant 1$, $p > -1$, $p + q > -1$ and $f \in L_{loc}$. Then conditions necessary and sufficient in order that, for some l', $f(x)g(x) \sim l'x^{p+q}(C, \mu)$ whenever $f(x) \sim lx^p (C, \lambda)$ for some l are that, for some a ($\geqslant 1$),

(i) $g \in L^\infty(1, a)$,

(ii) $$\frac{1}{u} \int_a^u t^{-q} g(t)dt = l_o - \frac{1}{\Gamma(\lambda+1)} \int_u^\infty (v-u)^\lambda d\,\alpha\,(v) \text{ for all } u > a,$$

where l_o is a constant and $\int_a^\infty t^\lambda\, |d\,\alpha\,(t)| < \infty$.

See (1) and (4), where Cesaro summability problems of a similar nature have been considered.

Theorem B is deduced from some theorems stated and proved in section 3.

2. Auxiliary Results

We first define the following subspaces of L_{loc} :

(i) $(C, \lambda, p, l) = \{f / f(x) \sim lx^p\ (C, \lambda)\}$ with the norm defined by

$$\| f \| = \sup_{t \geqslant 1} t^{-p-\lambda} | f_\lambda(t) | .$$

(ii) $C_\lambda = \{f / f \in (C, \lambda, 0, l) \text{ for some } l\}$.

(iii) $N_\lambda^a = \{f/f \in (C, \lambda, 0, 0), f(t) = 0 \text{ for } t < a, G \in L(a, \infty)\}$,

where $G(u) = uf(u) - \int_1^u f(t)dt$, with

$$\| f \| = \sup_{t \geqslant a} t^{-\lambda} |f_\lambda(t)| = \sup_{t \geqslant a} \left| \int_1^t u^{-\lambda-1} G_{\lambda-1}(u)du \right|$$

(iv) $(B, \lambda, p) = \{f/f(x) = 0(x^p)(C, \lambda)\}$

(v) $B_\lambda = (B, \lambda, 0)$

(vi) $C_0[a, \infty) = \{f/f \text{ is continuous for } t \geqslant a, f(t) \to 0 \text{ as } t \to \infty\}$ with $\| f \| = \sup_{t \geqslant a} |f(t)|$.

(vii) $BV[a, \infty) = \{f/f \text{ is of bounded variation in } [a, \infty)\}$.

The following lemmas will be used in the proofs of the theorems.

LEMMA 1 : If $(N_\lambda^a)^*$ denotes the dual space of N_λ^a, then every continuous linear functional $\Lambda \in (N_\lambda^a)^*$ is given by an equation of the form

$$\Lambda(f) = \frac{1}{\Gamma(\lambda)} \int_a^\infty f(u)du \int_u^\infty (t-u)^{\lambda-1} d\alpha(t), \text{ where}$$

$\int_a^\infty t^\lambda |d\alpha(t)| < \infty$, the norm of the functional Λ being given by

$$\| \Lambda \| = \frac{1}{\Gamma(\lambda)} \int_a^\infty t^\lambda |d\alpha(t)|.$$

Proof : Consider the equation $T_f(t) = y(t) = t^{-\lambda} \int_1^t (t-u)^{\lambda-1} f(u)du$ .... (2.1)

Then, $T_f \in C_0[a, \infty)$ whenever $f \in N_\lambda^a$.

Also, $\| T_f \|_{C_0(a, \infty)} = \sup_{t \geqslant a} |T_f(t)| = \| f \|_{N_\lambda^a}$ and thus the linear operator

$T : N_\lambda^a \to C_0[a, \infty)$ defined by (2.1) is a continuous linear operator which maps N_λ^a isometrically onto a subspace of $C_0[a, \infty)$. By the Riesz representation theorem, every $\Lambda \in (C_0[a, \infty))^*$ is given by

$$\Lambda(y) = \int_a^\infty y(t)d\beta(t), \; y \in C_0[a, \infty), \text{ where } \beta \in BV(a, \infty) \text{ and } \| \Lambda \| = \int_a^\infty |d\beta(t)|.$$

* See (4), Lemma 2.

Hence, by extending the functional $\Lambda(y)$, we can write every $\Lambda \in (N_\lambda^a)^*$ in the form

$$\Lambda(f) = \int_a^\infty (t^{-\lambda} \int_1^t (t-u)^{\lambda-1} f(u)du)\, d\beta(t), \beta \in BV\,(a, \infty). \text{ ------- } *$$

$$\text{i.e. } \Lambda(f) = \int_a^\infty f(u)du \int_u^\infty (t-u)^{\lambda-1} t^{-\lambda} d\beta(t)$$

$$= \frac{1}{\Gamma(\lambda)} \int_a^\infty f(u)du \int_u^\infty (t-u)^{\lambda-1}\, d\alpha(t) \text{ where}$$

$$\alpha(t) = -\Gamma(\lambda) \int_t^\infty u^{-\lambda} d\beta(u) \text{ and}$$

$$\| \Lambda \| = \int_a^\infty |d\beta(t)| = \frac{1}{\Gamma(\lambda)} \int_a^\infty t^\lambda \,|\, d\alpha\,(t)| \,.$$

Since $f \in N_\lambda^a$, the inversion of the order of integration is justified.

LEMMA 2 : Suppose $f.g \in B_\mu$ for some μ whenever $f \in C_\lambda$.

Then, (i) $g \in L^\infty(1, \infty)$,

(ii) There exist a ($\geqslant 1$) and K such that if

$$\Lambda(f) = \lim_{u \to \infty} \frac{1}{u} \int_1^u f(t)g(t)dt, \text{ then } \Lambda \in (N_\lambda^a)^* \text{ and } \| \Lambda \| \leqslant K.$$

Proof : The necessity of (i) follows trivially since constant functions belong to C_λ.

Now assume that (ii) is false, i.e. It is false that ' for some a, K, $|\Lambda(f)| \leqslant K \| f \|$ whenever $f \in N_\lambda^a$. .... (2.2)

$$\text{Now, for } \mu \geqslant 1, \frac{d}{du}(u^{-\mu} I_\mu f(u)g(u)) = u^{-\mu} I_{\mu-1} f(u)g(u) - \mu u^{-\mu-1} I_\mu f(u)g(u)$$

$$= u^{-\mu-1} G_{\mu-1(u)} \qquad \text{........... (2.3)}$$

where $G(u) = uf(u)g(u) - f(t)g(t)dt$.

We now define by induction an increasing sequence $\{a_n\}$ tending to $+\infty$ and a sequence of functions $\{f_n\}$ as follows :

Let $a_o = 1$ and suppose $a_1, \ldots . a_{n-1}$ and $f_1, \ldots . f_{n-1}$ have been defined such that $f_r \in N_\lambda^{r-1}$, $r = 1, \ldots n-1$.

Let $G_r(u) = uf_r(u)g(u) - \int_1^u f_r(t)g(t)dt$ for every r.

By (2.2) there exists $f_n \in N_\lambda^{a_{n-1}}$ such that

$$\| f_n \| < 2^{-n} \text{ and } \Lambda(f_n) > 1 \qquad \text{(2.4)}$$

$$\text{Let } a_n = 2a_{n-1} + \sum_{r=1}^{n} \int_1^\infty |G_r(u)| du \qquad \text{(2.5)}$$

Note that $\int_1^\infty |G_r(u)| du < \infty$ since $g \in L^\infty(1, \infty)$ and $f_r \in N_\lambda^{a_{r-1}}$.

Now define $f(t) = \sum_{r=1}^{\infty} f_r(t)$. Then $f(t) = 0$ for $t < 1$,

$f \in L_{loc}$ and $f(t) = \sum_{r=1}^{n} f_r(t)$ for $1 \leqslant t \leqslant a_n$.

Also,
$$\overline{\lim_{\substack{t_1\to\infty\\t_2\to\infty}}} |t_2^{-\lambda} f_\lambda(t_2) - t_1^{-\lambda} f_\lambda(t_1)| \leqslant \overline{\lim_{\substack{t_1\to\infty\\t_2\to\infty}}} \sum_{r=1}^{s} |t_2^{-\lambda}(fr)_\lambda(t_2) - t_1^{-\lambda}(fr)_\lambda(t_1)|$$
$$+ \overline{\lim_{\substack{t_1\to\infty\\t_2\to\infty}}} \sum_{r=s+1}^{\infty} |t_2^{-\lambda}(fr)_\lambda(t_2) - t_1^{-\lambda}(fr)_\lambda(t_1)|$$
$$\leqslant \overline{\lim_{\substack{t_1\to\infty\\t_2\to\infty}}} \sum_{r=s+1} \| f_r \| \leqslant 2 \sum_{r=s+1} 2^{-r} = 2^{1-s} \text{ for arbitary } s \in N.$$

Hence $t^{-\lambda} f_\lambda(t) \to$ a finite limit as $t \to \infty$, and thus the function f constructed belongs to C_λ.

$$\text{Now, } \int_1^{a_n} t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} G(u)du = \sum_{r=1}^{n} \int_1^{a_n} t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} Gr(u)du$$
$$= \sum_{r=1}^{n} \int_1^\infty t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} Gr(u)du - \sum_{r=1}^{n} \int_{a_n}^\infty t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} Gr(u)du$$
$$= \sum_{r=1}^{n} \int_1^\infty Gr(u)du \int_u^\infty (t-u)^{\mu-2} t^{-\mu-1} dt - \sum_{r=1}^{n} \int_{a_n}^\infty t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} Gr(u)du$$
$$= \sum_{r=1}^{n} \frac{\Gamma(\mu-1)}{\Gamma(\mu+1)} \Lambda(fr) - \sum_{r=1}^{n} \int_{a_n}^\infty t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} Gr(u)du$$

$$\text{But } \left| \sum_{r=1}^{n} \int_{a_n}^\infty t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} Gr(u)du \right| \leqslant \sum_{r=1}^{n} \int_{a_n}^\infty t^{-3} dt \int_1^\infty |Gr(u)| du$$
$$< \frac{1}{a_n} \sum_{r=1}^{n} \int_1^\infty |Gr(u)| du < 1 \text{ by (2.5).}$$

$$\text{Hence } \int_1^{a_n} t^{-\mu-1} dt \int_1^t (t-u)^{\mu-2} G(u)du > \frac{\Gamma(\mu-1)}{\Gamma(\mu+1)} n - 1 \text{ by (2.4)}$$

and by (2.3) it follows that $\int_1^{a_n} \frac{(a_n - u)^{\mu-1}}{a_n^{\mu}} f(u)g(u)du \to +\infty$ when $\mu > 1$.

contradicting the fact that $f.g \in B_\mu$ for some μ. Hence the necessity of (ii).

LEMMA 3 : If $p > -1$, $\lambda' > \lambda$, then $(C, \lambda, p, 1) \subset (C, \lambda', p, 1)$ and $(B, \lambda, p) \subset (B, \lambda', p)$.

This result is well known. Cf (2), Lemma 3.

LEMMA 4 : If $p > -1$, $p+q > -1$ and $g \in (C, \lambda, p, 1)$ [or (B, λ, p)], then $h \in (C, \lambda, p+q, 1)$ [or $(B, \lambda, p+q)$], where $h(x) = x^q g(x)$.

Cf. (2).Lemma 4.

LEMMA 5 : If $f \in B_\lambda$, $\lambda \geq 1$, then there exist constants H, K such that

(i) $\left|\int_1^t (t-u)^{\lambda-1} (v-u)^{\alpha} f(u)du\right| \leq H t^{\lambda} v^{\alpha}$ for $v \geq t$, $\alpha \geq 0$.

(ii) $\left|\int_1^t (t-u)^{\beta} [(v-u)^{\lambda-1} - v^{\lambda-1}] f(u)\, du\right| \leq K t^{\beta+1} (t^{\lambda-1} + v^{\lambda-1})$ for $v \geq t$, $\beta \geq 0$.

Proof : The results are trivial for $\lambda = 1$, and hence take $\lambda > 1$.

(i) Let $\lambda = n + p$ where $n \in IN_0$, $1 < p \leq 2$, and $M = \sup_{t \geq 1} t^{-\lambda} |f_\lambda(t)|$.

By partial integration we have

$$\int_1^t (t-u)^{\lambda-1} (v-u)^{\alpha} f(u)du = (-1)^{n+1} \int_1^t f_{n+1}(u) \left(\frac{\partial}{\partial u}\right)^{n+1} [(t-u)^{\lambda-1} (v-u)^{\alpha}]\, du$$

$$= \sum_{r=0}^{n+1} c_r J_r \text{ where } c_r \text{ is independent of t and v,} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.6)$$

and $J_r = \int_1^t (t-u)^{p+r-2} (v-u)^{\alpha-r} f_{n+1}(u)du$

$$= \frac{(t-1)^r (v-1)^{\alpha}}{(v-1)^r} \int_1^{b_r} (t-u)^{p-2} f_{n+1}(u)du, \text{ where } 1 \leq b_r \leq t,$$

by the Second Mean Value Theorem.

By Riesz's Mean Value theorem,

$$|J_r| \leq (t-1)^r (v-1)^{\alpha-r} \sup_{1 \leq b \leq b_r} \left|\int_1^b (b-u)^{p-2} f_{n+1}(u)du\right|$$

$$= (t-1)^r (v-1)^{\alpha-1} \sup_{1\leqslant b\leqslant b_r} |\Gamma(p-1) f'_{n+p}(b)| \leqslant \Gamma(p-1) M t^{\lambda} v^{\alpha},$$

since $\left[\dfrac{(t-1)}{(v-1)}\right]^r < 1$, and λ, α are non-negative.

Hence, (2.6) gives (i)

(ii) Take $\beta = n+p$ where $n \in$ INo, $0 < p \leqslant 1$. As before, we have

$$\int_1^t (t-u)^{\beta} [(v-u)^{\lambda-1} - v^{\lambda-1}] f(u)du$$

$$= (-1)^{n+1} \int_1^t f_{n+1}(u) \left(\frac{\partial}{\partial u}\right)^{n+1} \left\{(t-u)^{\beta} [v-u)^{\lambda-1} - v^{\lambda-1}]\right\} du \quad \ldots\ldots\ (2.7)$$

But, $\left(\frac{\partial}{\partial u}\right)^{n+1} \left\{(t-u)^{\beta} [v-u)^{\lambda-1} - v^{\lambda-1}]\right\} = Co\,(t-u)^{\beta-n-1} [(v-u)^{\lambda-1} - v^{\lambda-1}]$

$$+ \sum_{r=1}^{n+1} C_r (t-u)^{\beta-n-1+r} (v-u)^{\lambda-1-r} \quad \ldots\ldots\ (2.8)$$

As in (i), $\left|\int_1^t (t-u)^{\beta-n-1+r} (v-u)^{\lambda-1-r} f_{n+1}(u)du\right| \leqslant \Gamma(\beta-n) M t^{\beta+1} \nu^{\lambda-1}$

$\ldots\ldots\ (2.9)$

Also, $\left|\int_1^t (t-u)^{\beta-n-1} [(v-u)^{\lambda-1} - v^{\lambda-1}] f_{n+1}(u)du\right|$

$$= [v^{\lambda-1} - (v-u)^{\lambda-1}] \left|\int_{\eta}^t (t-u)^{\beta-n-1} f_{n+1}(u)\, du\right| \text{ where } 1 \leqslant \eta \leqslant t$$

$$\leqslant 2\Gamma(\beta-n) M t^{\beta+1} [t^{\lambda-1} + (\lambda-1) t v^{\lambda-1}]$$

(2.7), (2.8), (2.9) and (2.10) give the required result.

3. Theorems and their Proofs

THEOREM 1: If $f.g \in B_{\mu}$ for some u whenever $f \in C_{\lambda}$, then there exists a $(\geqslant 1)$ such that

(i) $g \in L^{\infty}(1, a)$

(ii) $\dfrac{1}{u}\displaystyle\int_1^u g(t)dt = 1_o \dfrac{1}{\Gamma(\lambda+1)} \int_u^{\infty} (v-u)^{\lambda} d\alpha(v)$ for all $u > a$,

where 1_o is a constant, and $\int_a^{\infty} v^{\lambda} |d\alpha(v)| < \infty$.

Proof : The necessity of (i) follows from Lemma 2 (i).

By Lemma 2, there exist a_0 and K such that

$$\lim_{u\to\infty} \left|\frac{1}{u}\int_1^u f(t)g(t)dt\right| \leqslant K\|f\| \text{ whenever } f \in N_\lambda^{a_0} \quad \ldots\ldots\ldots\ldots (3.1)$$

Also, if $f \in N_\lambda^{a_0}$, then $\frac{1}{u}\int_1^u |f(t)|\,dt \in V(a_0,\infty)$. $\quad\ldots\ldots\ldots\ldots$ (3.2)

Now, (3.1) implies that there exists $a \geqslant a_0$ such that if

$$\Lambda_u(f) = \frac{1}{u}\int_1^u f(t)g(t)dt, \text{ then, whenever } u > a,\ \Lambda_u \in (N_\lambda^a)^* \quad \ldots\ldots\ldots (3.3)$$

For, if (3.3) is false, by the method used in Lemma 2, we can construct

$\{b_n\}\Uparrow$, $b_n \to +\infty$ and $f \in N_\lambda^{a_0}$ such that $\frac{1}{b_n}\int_1^{b_n} f(t)g(t)dt > n$,

contradicting the fact that $\frac{1}{u}\int_1^u f(t)g(t)dt$ is bounded whenever $f \in N_\lambda^{a_0}$, which is a consequence of (3.2) and $g \in L^\infty(1,\infty)$.

Hence, (3.3) and Lemma 1 give : Whenever $u > a$,

$$\frac{1}{u}\int_1^u f(t)g(t)dt = \frac{1}{\Gamma(\lambda)}\int_a^\infty f(u)du\int_u^\infty (t-u)^{\lambda-1}\,d\alpha(t), \text{ where}$$

$$\int_a^\infty t^\lambda\,|d\alpha(t)| < \infty, \text{ for } f \in N_\lambda^a. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

Clearly, the function $\chi_{(a,u)}$ belongs to N_λ^a. Hence (3.4) gives

$$\frac{1}{u}\int_a^u g(t)dt = \frac{1}{\Gamma(\lambda)}\int_a^u dt\int_t^\infty (v-t)^{\lambda-1}\,d\alpha(v) \text{ for all } u > a.$$

Since $\int_a^\infty t^\lambda |\,d\alpha(t)\,| < \infty$, $\frac{1}{\Gamma(\lambda)}\int_a^u dt\int_t^\infty (v-t)^{\lambda-1}\,d\alpha(v)$

$$= \frac{1}{\Gamma(\lambda)}\int_a^\infty dt\int_t^\infty (v-t)^{\lambda-1}\,d\alpha(v) - \frac{1}{\Gamma(\lambda)}\int_u^\infty dt\int_t^\infty (v-t)^{\lambda-1}\,d\alpha(v)$$

$$= l_0 - \frac{1}{\Gamma(\lambda+1)}\int_a^\infty (v-u)^\lambda\,d\alpha(v), \text{ where } l_0 = \frac{1}{\Gamma(\lambda+1)}\int_a^\infty (v-a)^\lambda\,d\alpha(v).$$

Hence the result,

THEOREM 2 : If $f \in B_\lambda$ and (i) $h \in L^\infty(1, a)$

(ii) $$\frac{1}{u}\int_a^u h(t)dt = \frac{1}{\Gamma(\lambda+1)}\int_u^\infty (v-u)^\lambda \, d\,\alpha(v), \text{ where} \int_a^\infty v^\lambda |d\alpha(v)| < \infty, \text{ then}$$

$$\text{I: } \overline{\lim_{t\to\infty}}\, t^{-\lambda} I_\lambda f(t)h(t) = -\overline{\lim_{t\to\infty}} \frac{t^{-1}}{\Gamma(\lambda)}\int_a^t I_\lambda (vf(v) - f_1(v))d\,\alpha(v).$$

Proof : Since $f \in B_\lambda$, by Lemma 4 we have

$I_\lambda (vf(v) - f_1(v)) = o(v^{\lambda+1})$, and hence the R.H.S. of I exists, since $\int_a^\infty v^\lambda |d\alpha(v)| < \infty$.

Now (ii) gives $$h(u) = \frac{1}{\Gamma(\lambda+1)}\int_u^\infty (v-u)^\lambda \, d\alpha(v) - \frac{u}{\Gamma(\lambda)}\int_u^\infty (v-u)^{\lambda-1} \, d\alpha(v)$$

for $u > a$.

$$\text{Hence } t^{-\lambda} I_\lambda f(t)h(t) = t^{-\lambda}\int_1^a \frac{(t-u)^{\lambda-1}}{\Gamma(\lambda)} f(u)h(u)du + I_1 + I_2 \qquad \ldots\ldots\ (3.5)$$

$$\text{where } I_1 = \frac{t^{-\lambda}}{\Gamma(\lambda)}\left\{\frac{1}{\Gamma(\lambda+1)}\int_a^t (t-u)^{\lambda-1} f(u)du \int_u^t (v-u)^\lambda d\alpha(v) - \frac{1}{\Gamma(\lambda)}\int_a^t (t-u)^{\lambda-1} uf(u)du \int_u^t (v-u)^{\lambda-1} d\alpha(v)\right\}$$

$$= \frac{t^{-\lambda}}{\Gamma(\lambda)}\left\{\frac{1}{\Gamma(\lambda+1)}\int_a^t d\alpha(v)\int_a^v (v-u)^\lambda (t-u)^{\lambda-1} f(u)du - \frac{1}{\Gamma(\lambda)}\int_a^t d\alpha(v)\int_a^v (v-u)^{\lambda-1}(t-u)^{\lambda-1} uf(u)du\right\} \qquad \ldots\ldots\ (3.6)$$

$$\text{and } I_2 = \frac{t^{-\lambda}}{\Gamma(\lambda)}\left\{\frac{1}{\Gamma(\lambda+1)}\int_a^t (t-u)^{\lambda-1} f(u)du\int_t^\infty (v-u)^\lambda d\,\alpha(v) - \frac{1}{\Gamma(\lambda)}\int_a^t (t-u)^{\lambda-1} f(u)du\int_t^\infty (v-u)^{\lambda-1} d\alpha(v)\right\}$$

$$= \frac{t^{-\lambda}}{\Gamma(\lambda)}\left\{\left(\frac{1}{\Gamma(\lambda+1)} + \frac{1}{\Gamma(\lambda)}\right)\int_t^\infty d\alpha(v)\int_a^t (t-u)^{\lambda-1}, (v-u)^\lambda f(u)du.\right.$$

$$-\frac{1}{\Gamma(\lambda)}\int_t^{\infty} d\alpha(v)\int_a^t v(t-u)^{\lambda-1}\,(v-u)^{\lambda-1} f(u)du\Big\}$$

Hence, by Lemma 5(i) we get

$$|I_2| \leqslant K_1 t^{-\lambda}\int_t^{\infty} t^{\lambda} v^{\lambda}\,|d\alpha(v)| \to 0 \text{ as } t\to\infty \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

Now, by (3.6) we get $I_1 - \frac{t^{-1}}{\Gamma(\lambda)}\int_a^t I_\lambda\,(f_1(v) - vf(v)d\alpha(v)$

$$= \frac{t^{-\lambda}}{\Gamma(\lambda)}\Big\{\frac{1}{\Gamma(\lambda+1)}\int_a^t d\alpha(v)\int_a^v (v-u)^{\lambda}\left[(t-u)^{\lambda-1} - t^{\lambda-1}\right] f(u)du$$

$$-\frac{1}{\Gamma(\lambda)}\int_a^t d\alpha(v)\int_a^v (v-u)^{\lambda-1}\left[(t-u)^{\lambda-1} - t^{\lambda-1}\right] uf(u)\,du\Big\}$$

Hence $\left| I_1 - \frac{t^{-1}}{\Gamma(\lambda)}\int_a^t I_\lambda\,(f_1(v) - vf(v))\,d\alpha(v)\right|$

$$\leqslant K_2 t^{-\lambda}\int_a^t v^{\lambda+1}\,(v^{\lambda-1} + t^{\lambda-1})|d\alpha(v)| \qquad \text{by Lemma 5 (ii)}$$

$$\leqslant K_2 t^{-\lambda}\int_a^w v^{\lambda+1}\,(v^{\lambda-1} + t^{\lambda-1})\,|d\alpha(v)| + 2K_2\int_w^{\infty} v^{\lambda}\,|d\alpha(v)| \text{ ,where } a<w<t, \text{ and}$$

$$\text{Hence } \overline{\lim_{w\to\infty}}\ \overline{\lim_{t\to\infty}}\left|I_1 - \frac{t^{-1}}{\Gamma(\lambda)}\int_a^t I_\lambda\,(f_1(v) - vf(v))d\alpha(v)\right| = 0 \quad \ldots\ldots\ldots\ldots (3.8)$$

Now, (i) implies that $t^{-\lambda}\int_1^a \frac{(t-u)^{\lambda-1}}{\Gamma(\lambda)} f(u)h(u)du\ 0$ as $t\to\infty$.

Hence, (3.5), (3.7) and (3.8) give I.

THEOREM 3 : Conditions necessary and sufficient in order that $f.g \in C_\lambda$ whenever $f \in C_\lambda$ are that, for some a ($\geqslant$ 1)

(i) $g \in L^{\infty}(1, a)$,

(ii) $\frac{1}{u}\int_a^u g(t)dt = l_0 - \frac{1}{\Gamma(\lambda+1)}\int_a^{\infty}(v-u)^{\lambda}\,d\alpha(v)$ for all $u \geqslant a$

where l_0 is a constant and $\int_a^{\infty} t^{\lambda}|d\alpha(t)| < \infty$.

Proof : If $f.g \in C_\lambda$ whenever $f \in C_\lambda$, then $f.g \in B_\lambda$ whenever $f \in C_\lambda$, and by Theorem 1, (i) and (ii) are necessary.

If (i) and (ii) hold, then $g(u) = l_o + h(u)$, where $h(u)$ is as in Theorem 2.

Hence $t^{-\lambda} I_\lambda f(t)g(t) = l_o t^{-\lambda} f_\lambda(t) + t^{-\lambda} I_\lambda f(t)h(t) \rightarrow$ a finite limit as $t \rightarrow \infty$ whenever $f \in C_\lambda$, by Theorem 2.

THEOREM 4 : If $\mu \geqslant \lambda \geqslant 1$, $p+q > -1$, $p > -1$ then conditions necessary and sufficient in order that, for some l',

$f.g \in (C, \mu, p+q, l')$ whenever $f \in (C, \lambda, p, l)$ for some l are that for some $a(\geqslant 1)$,

(i) $g \in L^\infty(1, a)$

(ii) $\frac{1}{u}\int_a^u t^{-q} g(t)dt = l_o - \frac{1}{\Gamma(\lambda+1)}\int_u^\infty (v-u)^\lambda \, d\alpha(v)$ for all $u > a$,

where l_o is constant, $\int_a^\infty t^\lambda |d\alpha(t)| < \infty$.

Proof : Since $p > -1$, $p+q > -1$ we can consider $x^{-p}f(x)$ and $x^{-q}g(x)$ instead of $f(x)$ and $g(x)$ in the previous theorem, and use Lemma 4.

Hence Theorem 1 gives the necessity of (i) and (ii).

Again by Theorem 3, (i), (ii) and $f \in (C, \lambda, p, 1)$ imply that $f.g \in (C, \lambda, p+q, l')$ for some l', and hence $f.g \in (C, \mu, p+q, l')$, by Lemma 3.

This completes the proof.

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