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# A RESULT ON TWO-DIMENSIONAL POLAR LATTICES

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**Abstract** : Suppose P and $P^o$ denote closed positive and open positive quadrants in $R^2$ respectively. Let $\Lambda$ be any lattice in $R^2$ with polar lattice $\Lambda^*$. Let F be a convex and symmetric (with respect to the axes of coordinates) distance function with F(1,0) = F(0,t) = 1, where t $\in$ R and let $\mu$ = area ($\underline{x} \in R^2 / F(\underline{x}) \leqslant 1$). For certain distance functions F, there exist non-zero $\underline{x} \in P \cap \Lambda$ and $\underline{y} \in P^o \cap \Lambda^*$ such that $\mu F(\underline{x}) F(\underline{y}) \leqslant \gamma_t$, where $\gamma_t$ is a constant depending on t and the distance function. There exist a lower bound 2(t + 1/t) and an upper bound 4(t + 1/t) for $\gamma_t$ over all convex symmetric distance functions.

## 1. Introduction

Let P and $P^o$ denote closed positive and open positive quadrants in $R^2$ respectively. Let $\Lambda$ be any lattice in $R^2$ with polar lattice $\Lambda^*$. Let F be a convex and symmetric (with respect to the axes of coordinates) distance function with F(1,0) = F(0,t) = 1, where t $\in$ R. Without loss of generality we can take $t \geqslant 1$. If $t \leqslant 1$, we have the same situation as in the case when $t \geqslant 1$ with the coordinate axes interchanged. Let $\mu = \text{area}\{\underline{x} \in R^2 / f(\underline{x}) \leqslant 1\}$.

Hossain and Worley[3] have shown that for certain distance functions F, there exist non-zero $\underline{x} \in P \cap \Lambda$ and $\underline{y} \in P^o \cap \Lambda^*$ such that

$$\mu F(\underline{x}) F(\underline{y}) \leqslant \gamma_t,$$

where $\gamma_t$ is a constant depending on t and the distance function. In this note we show that $\gamma_t$ has a lower bound and an upper bound over all the convex symmetric distance functions. In this note symmetric means the symmetricity with respect to the axes of coordinates.

## 2. Discussion

The following notations will be used frequently in this section.

$$F_1(\Lambda) = \inf\{F(\underline{x}) : \underline{x} \in \Lambda \cap P\}$$
$$F_2(\Lambda^*) = \inf\{F(\underline{x}^*) : \underline{x}^* \in \Lambda^* \cap P^o\},$$

where F is a distance function.

**Theorem:**

If F is any convex symmetric distance function with F(1,0)=F(0,t)=1, then

$$2(t+1/t) \leqslant \mu\, F_1(\Lambda)\, F_2(\Lambda^*) \leqslant 4(t+1/t)$$

for the lattice $\Lambda$ with basis $\{(1,0), (0,t)\}$

The lower bound is best possible for the distance function $F(x_1, x_2) = |x_1| + 1/t\,|x_2|$ and the lattice $\Lambda$ with a basis (1,0) and (0,t). The upper bound may not be best possible, but cannot be below 4t. In order to prove the theorem, we use the following lemmas.[2]

**Lemma 1**

Let $\Lambda$ be the lattice with a basis $\{(1,0), (0,t)\}$. Then $\min\limits_{\alpha} \mu F_1(\Lambda)\, F_2(\Lambda^*) = 2(t+1/t)$, for the convex symmetric polygonal distance function F given by

$$F(x_1,x_2) = \max\left\{\frac{1-\alpha}{\alpha}|x_1| + 1/t\,|x_2|,\ |x_1| + \frac{1-\alpha}{\alpha t}|x_2|\right\}$$

where $1/2 \leqslant \alpha \leqslant 1$.

($\alpha$ has to satisfy the above conditions since F is convex and symmetric).

**Lemma 2**

Let $\Lambda$ be a lattice with basis $\{(1,0),(0,t)\}$. Let F be the convex polygonal distance function, where $F(x_1, x_2) = 1$ has two more vertices at $(\alpha,\alpha t)$ and $(\beta,\beta/t)$ in $P^o$ in addition to (1,0) and (0,t), where $1/2 \leqslant \alpha \leqslant 1$ and the limit of $\beta$ depends on $\alpha$.

$$\text{Then } \min_{\alpha,\beta} \mu\, F_1(\Lambda)\, F_2(\Lambda^*) \geqslant 2(t+1/t).$$

From Lemma 1 and Lemma 2, we can establish the left hand side of the inequality in the theorem.

Suppose $F(x_1,x_2) = 1$ intersects the lines OL at B and OM at C respectively, where $L \equiv (1,t)$ and $M = (1,1/t)$.

Let $B \equiv (\alpha,\alpha t)$ and $C \equiv (\beta,\beta/t)$.

The curve $F(x_1,x_2) = 1$ passes through the points $A \equiv (0,t)$ and $D \equiv (1,0)$.

Then $F_1(\Lambda) = 1$.

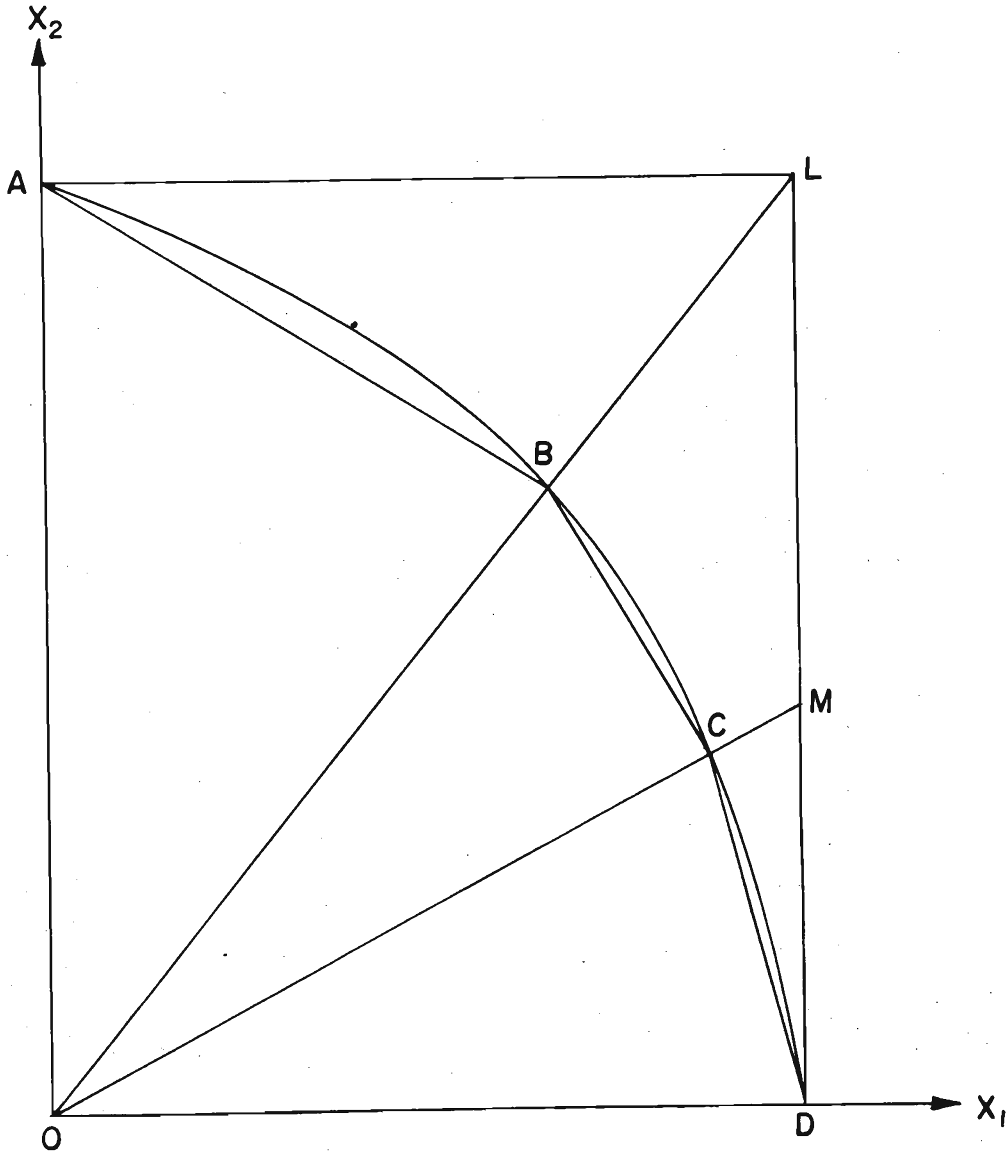


Figure 1. The curve $F(x_1,x_2) = 1$ passes through the points A = (0,t) and D = (1,0). It intersects the lines OL at B and OM at C respectively, where L = (1,t) and M = (1, 1/t).

and $F_2(\Lambda^*) = F(1,1/t) = 1/\beta\ F(\beta,\beta/t) = 1/\beta$.

Let $G(x_1, x_2) = 1$ be the equation of the polygonal arc ABCD. Then for this distance function G,

$$G_1(\Lambda) = 1 \text{ and } G_2(\Lambda^*) = 1/\beta$$

let $\mu_G = 4 \times \text{area}\{\text{polygon OABCD}\}$
Then from Lemma 2 we have

$$\mu_g\, G_1(\Lambda)\ G_2(\Lambda^*) = \mu_G.1/\beta \geqslant 2(t + 1/t) \text{ for all suitable } \alpha \text{ and } \beta.$$

Now let $\mu = \text{area}\{\underline{x} \in R^2 : F(\underline{x}) \leqslant 1\}$.

Then from the convexity $\mu \geqslant \mu_G$.

Hence $\mu F_1(\Lambda)\ F_2(\Lambda^*) = \mu.\, 1/\beta \geqslant \mu_G.\, 1/\beta \geqslant 2(t+1/t)$.
ie we have proved that

$\mu F_1(\Lambda)\ F_2(\Lambda^*) \geqslant 2(t+1/t)$ for all convex symmetric distance functions F and the lattice $\Lambda$ with basis (1,0) and (0,t).

$\therefore \min\limits_{F} \mu F_1(\Lambda) F_2(\Lambda^*) \geqslant 2(t+1/t)$.

Now we proceed to prove the right hand side of the inequality in the theorem.

Let $\Lambda$ be any two dimensional lattice and $(a,b) \in \Lambda \cap P^o$ and $(-c,d) \in \Lambda \cap Q$ be two points such that F(a,b) and F(d,c) are minimal, where Q is the open second quadrant.

Note that if $(-c,d) \in \Lambda$, then $\frac{(d,c)}{d(\Lambda)} \in \Lambda^*$, where $d(\Lambda)$ is the determinant of $\Lambda$.

Let $D = \{(x_1, x_2) \in R^2 : F(x_1, x_2) \leqslant F(a,b),\ F(-x_2, x_1) \leqslant F(d,c)\}$.

Then from the choice of (a,b) and (−c,d) there are non-zero points of $\Lambda$ in D.

Then there are no non-zero points of $\Lambda$ in the parallelogram.

$\{(x_1, x_2) \in R^2 : |-x_1 + tx_2| \leqslant tF(d,c), |tx_1+x_2| \leqslant t\,F(a,b)\}$
which lies entirely in D.

Hence by Minkowski's linear form theorem[1], we have

$$t^2\, F(a,b)\, F(d,c) \leqslant (1+t^2)\, d(\Lambda).$$

$$\therefore\ F(a,b)\ \frac{F(d,c)}{d(\Lambda)} \leqslant 1 + 1/t^2.$$

Now $\mu = \text{area}\left\{(x_1, x_2) \in R^2 : F(x_1, x_2) \leqslant 1\right\}$

$\leqslant 4t$, as $F(x_1, x_2)$ is convex and symmetric.

$$\therefore\ \mu\ F_1(\Lambda)\ F_2(\Lambda^*) \leqslant 4(t+1/t).$$

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