RESEARCH ARTICLE

Radii problems and some other properties of certain classes of analytic functions with boundary rotation

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Submitted: 28 September 2020; Revised: 07 January 2021; Accepted 28 February 2021

Abstract: In this work, some generalized classes of convex, close-to-convex, quasi-convex and strongly convex functions $f(z)$ in the frame of a unit disc U are introduced. The first task is to discuss some geometric properties of a class of functions $f(z)$ for which $f(U)$ has a boundary rotation of at most $m\pi$, $m \geq 2$. Furthermore, the smallest disc of the linear combinations of functions belonging to the classes $\mathcal{V}_m[A, B]$, $CCV_{m_1 m_2}[A, B, C, D]$ and $QCV_{m_1 m_2}[A, B, C, D]$. Condition for univalence, covering theorem and distortion result of a more generalized strongly close-to-convex functions are also obtained. Meanwhile, some remarkable cases and the consequences of our investigation are also highlighted.

Keywords: Analytic function, Univalent function, Strongly close-to-convex function, Subordination.

INTRODUCTION

Theory on geometric functions is the part of the complex analysis, which considers the geometric characterization of analytic functions and established around the turn of the twentieth century. Regardless of the well-known coefficient problem, the Bieberbach conjecture that was explained by Louis de Branges in 1984 recommends different methodologies and bearings of concentrates in the field of Univalent function theory. As a result,

several authors paid their attention to establishing certain subclasses of analytic functions defined by differential subordination, such as the classes of starlike, convex, close-to-convex and quasi-convex functions (Duren, 2001; Goodman, 1983).

One natural extension of these classes is the class $\mathcal{V}_m$ ($m \geq 2$) of functions with bounded boundary rotation. Löwner (1917) started work related to this topic but Paatero (1931) systematically developed its properties. This direction of research has experienced many advancements and contributions, see (Brannan, 1969; Noor & Thomas, 1980; Noor, 1981; 1988; 2002; 2009; Noor & Malik, 2010; Saliu, 2019; Afis & Noor, 2020; Saliu & Noor, 2020). Following the aforementioned works and taking into account the open unit disc U , the aim of the present study is to introduce the novel generalizations of convex, close-to-convex, quasi-convex and strongly convex functions. Then the smallest disc of the linear combinations of functions belonging to these classes was found. Moreover, the geometric features of a class of functions for which the image domain has boundary rotation under certain conditions was investigated. A more general concept of strongly close-to-convex functions is also initiated. In this direction, conditions for the univalence, covering theorem and distortion results for this novel class are obtained.

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METHODOLOGY

Some mathematical preliminaries and definitions that will play a key role in proving the main results are presented here.

Let $\mathcal{H}$ denotes the class of functions $f(z)$ in the open unit disk U with the series form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad \dots(1)$$

If $f(z)$ and $g(z)$ are analytic functions in U , then $f(z)$ is subordinate to $g(z)$ (written as $f(z) \prec g(z)$) if there exist a Schwarz function $w(z)$ such that $f(z) = g(w(z))$, $z \in U$.

Janowski (1973) introduced the class $\mathcal{P}[\mathcal{A}, \mathcal{B}]$, $-1 \leq \mathcal{B} < \mathcal{A} \leq 1$ of functions $p(z)$ satisfying the inequality

$$\left| p(z) - \frac{1 - \mathcal{A}\mathcal{B}r^2}{1 - \mathcal{B}^2 r^2} \right| \leq \frac{(\mathcal{A} - \mathcal{B})r}{1 - \mathcal{B}^2 r^2}, \quad 0 \leq r < 1. \quad \dots(2)$$

Also, the class $\mathcal{P}^\lambda[\mathcal{A}, \mathcal{B}]$ was introduced by Raina *et al.* (2020) and defined it as

$$\left| p^{\frac{1}{\lambda}}(z) - \frac{1 + \mathcal{A}\mathcal{B}r^2}{1 - \mathcal{B}^2 r^2} \right| \leq \frac{(\mathcal{A} - \mathcal{B})r}{1 - \mathcal{B}^2 r^2}, \quad 0 \leq r < 1, \quad 0 < \lambda \leq 1. \quad \dots(3)$$

As a special case, $\mathcal{P}^\lambda[\mathcal{A}, \mathcal{B}]$ reduces to $\mathcal{P}[\mathcal{A}, \mathcal{B}]$ for $\lambda = 1$. Similar to the class $\mathcal{P}_m$, $m \geq 2$ introduced by Pinchuk (1971), Noor (1991a) defined the class $\mathcal{P}_m[\mathcal{A}, \mathcal{B}]$ and used it to find the radius of convexity and starlikeness for the classes $\mathcal{V}_m[\mathcal{A}, \mathcal{B}]$ and $\mathcal{R}_m[\mathcal{A}, \mathcal{B}]$ respectively. For more details regarding these classes, refer to (Noor, 1988, 1991b, 1994; 2009; Noor & Malik, 2010).

We define the following classes of functions as follows:

Definition 2.1. Let $f(z)$ be of the form (1) and $m_1, m_2 \geq 2$, $-1 \leq \mathcal{B} < \mathcal{A} \leq 1$, $-1 \leq \mathcal{C} < \mathcal{D} \leq 1$. Then we say $f \in \mathcal{CCV}_{m_1 m_2}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ if there exists a function $g \in \mathcal{V}_{m_1}[\mathcal{A}, \mathcal{B}]$ such that

$$\frac{f'(z)}{g'(z)} \in \mathcal{P}_{m_2}[\mathcal{C}, \mathcal{D}]. \quad \dots(4)$$

Definition 2.2. Let $f(z)$ be of the form (1) and

$m_1, m_2 \geq 2$, $-1 \leq \mathcal{B} < \mathcal{A} \leq 1$, $-1 \leq \mathcal{C} < \mathcal{D} \leq 1$. Then we say $f \in \mathcal{QCV}_{m_1 m_2}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ if there exists a function $g \in \mathcal{V}_{m_1}[\mathcal{A}, \mathcal{B}]$ such that

$$\frac{(zf'(z))'}{g'(z)} \in \mathcal{P}_{m_2}[\mathcal{C}, \mathcal{D}]. \quad \dots(5)$$

Definition 2.3. Let $f(z)$ be of the form (1) and $m \geq 2$, $-1 \leq \mathcal{B} < \mathcal{A} \leq 1$, $-1 \leq \mathcal{C} < \mathcal{D} \leq 1$. Then we say $f \in \mathcal{K}^\lambda_m[\mathcal{A}, \mathcal{B}]$ if there exists a function $g \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$ such that

$$\frac{f'(z)}{g'(z)} \in \mathcal{P}^\lambda[\mathcal{A}, \mathcal{B}], \quad 0 < \lambda \leq 1. \quad \dots(6)$$

By specifying certain values of the underlying parameters in the above definitions, we obtain the following sub-classes, which have been studied by many researchers:

- (i) $\mathcal{CCV}_{22}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$, is the well-known subclass of close-to-convex function introduced by Silvia(1983).
- (ii) For $\mathcal{C} = \mathcal{A}$, $\mathcal{D} = \mathcal{B}$ and $m_1 = m_2 = m$, we obtain the class $\mathcal{T}_m[\mathcal{A}, \mathcal{B}]$ that was studied by Noor (1991b).
- (iii) $\mathcal{CCV}_{22}[1, -1, 1, -1] \equiv \mathcal{CCV}$, is the usual well-known class of close-to-convex functions introduced and examined by Kaplan(1952).
- (iv) For $\mathcal{C} = \mathcal{A}$, $\mathcal{D} = \mathcal{B}$ and $m_1 = m_2 = 2$, we get the class $\mathcal{QCV}[\mathcal{A}, \mathcal{B}]$ that was explored in (Aliintaş & Kiliç, 2018).
- (v) $\mathcal{QCV}[1, -1, 1, -1]$ is the well-known class of quasi convex functions that was first introduced and examined in (Noor,1980).
- (vi) For $\lambda = 1$, $m = 2$, $\mathcal{K}^\lambda_m[\mathcal{A}, \mathcal{B}]$ becomes the class of functions considered by Raina *et al.* (2020).
- (vii) $\mathcal{K}^\lambda_m[1, -1] = \mathcal{K}^\lambda_m$, is the class explored by Noor (2002).

To establish the main results, the following results are required.

Lemma 2.1.(Stump, 1971) If $|u - a| \leq d$ and $|v - a| \leq d$, where a and d are real with $a > d \geq 0$, and

$$w = u \frac{1}{1 + \lambda e^{i\alpha}} + v \frac{1}{1 + \lambda^{-1} e^{-i\alpha}}, \quad \dots(7)$$

where α is real, $\lambda > 0$ and $\alpha \in [0, \pi)$, then

$$\operatorname{Re} w \geq a - d \sec \frac{\alpha}{2}. \quad \dots(8)$$

The following lemma exhibits some practice related

to our consequences.

Lemma 2.2. Let $p \in \mathcal{P}_m[\mathcal{A}, \mathcal{B}]$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \begin{cases} 1 + \frac{m^2(\mathcal{B}-\mathcal{A})^2}{4\mathcal{B}^2} \left(\frac{\mathcal{B}^2 r^2}{1-\mathcal{B}^2 r^2} \right), & \mathcal{B} \neq 0 \\ 1 + \frac{m^2 \mathcal{A}^2 r^2}{4}, & \mathcal{B} = 0 \end{cases} \quad \dots(9)$$

$$\int_0^{2\pi} |p(z)|^2 d\theta \leq \begin{cases} r + \frac{m^2(\mathcal{B}-\mathcal{A})^2}{8\mathcal{B}^3} \left[\log \left(\frac{1+\mathcal{B}r}{1-\mathcal{B}r} \right) - 2\mathcal{B}r \right], & \mathcal{B} \neq 0 \\ r + \frac{m^2 \mathcal{A}^2 r^3}{12}, & \mathcal{B} = 0 \end{cases} \quad \dots(10)$$

Proof. Since $p \in \mathcal{P}_m[\mathcal{A}, \mathcal{B}]$, then

$$p(z) = \frac{m+2}{4} p_1(z) - \frac{m-2}{4} p_2(z),$$

where

$$p_i(z) \prec \frac{1+\mathcal{A}z}{1+\mathcal{B}z}, \quad i = 1, 2 \quad \dots(11)$$

and by subordination property,

$$|p(z)| \leq \max_{|z| \leq r} \left| \frac{1+\mathcal{A}z}{1+\mathcal{B}z} \right|.$$

$$\text{Let } p_i(z) = 1 + \sum_{n=1}^{\infty} d_{i,n} z^n \quad \dots(12)$$

and we can write

$$\frac{1+\mathcal{A}z}{1+\mathcal{B}z} = 1 + \sum_{n=1}^{\infty} \vartheta_n z^n, \quad \text{where}$$

$$\vartheta_n = (-1)^n \mathcal{B}^{n-1} (\mathcal{B} - \mathcal{A}), \quad n = 1, 2, 3, \dots \quad \dots(13)$$

Thus, by Parseval's identity and subordination property, we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta &\leq 1 + \sum_{n=1}^{\infty} \left| \left(\frac{m+2}{4} \right) d_{1,n} - \left(\frac{m-2}{4} \right) d_{2,n} \right|^2 r^{2n} \\ &\leq 1 + \sum_{n=1}^{\infty} \left[\left(\frac{m+2}{4} \right) |d_{1,n}| + \left(\frac{m-2}{4} \right) |d_{2,n}| \right]^2 r^{2n} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{m^2}{4} |\vartheta|^2 r^{2n} \\ &= 1 + \frac{m^2 (\mathcal{B}-\mathcal{A})^2}{4\mathcal{B}^2} \sum_{n=1}^{\infty} \mathcal{B}^{2n} r^{2n} \\ &= 1 + \frac{m^2 (\mathcal{B}-\mathcal{A})^2}{4\mathcal{B}^2} \left(\frac{\mathcal{B}^2 r^2}{1-\mathcal{B}^2 r^2} \right). \end{aligned} \quad \dots(13)$$

Using (14), we obtain

$$\begin{aligned} \frac{1}{2\pi} \int_0^r \int_0^{2\pi} |p(z)|^2 d\theta d\rho &\leq r + \frac{m^2 (\mathcal{B}-\mathcal{A})^2}{4\mathcal{B}^2} \int_0^r \left(\frac{1}{1-\mathcal{B}^2 \rho^2} - 1 \right) d\rho \\ &= r + \frac{m^2 (\mathcal{B}-\mathcal{A})^2}{8\mathcal{B}^3} \left[\log \left(\frac{1+\mathcal{B}r}{1-\mathcal{B}r} \right) - 2\mathcal{B}r \right]. \end{aligned}$$

If $\mathcal{B} = 0$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta &\leq 1 + \left[\left(\frac{m+2}{4} \right) \mathcal{A} + \left(\frac{m-2}{4} \right) \mathcal{A} \right]^2 r^2 \\ &= 1 + \frac{m^2}{4} \mathcal{A}^2 r^2 \end{aligned}$$

and

$$\frac{1}{2\pi} \int_0^r \int_0^{2\pi} |p(z)|^2 d\theta d\rho \leq r + \frac{m^2}{12} \mathcal{A}^2 r^3.$$

Remark 2.1. As a consequence of Lemma 2.2, we have that

- (i) for $m = 2$, the result coincide with the one obtained by Cho & Kumar (2019),
- (ii) for $p \in \mathcal{P}_m[1, -1]$,

$$\frac{1}{2\pi} \int_0^r \int_0^{2\pi} |p(z)|^2 d\theta d\rho \leq \frac{1 + (m^2 - 1)r^2}{1 - r^2} \quad \dots(15)$$

and

$$\frac{1}{2\pi} \int_0^r \int_0^{2\pi} |p(z)|^2 d\theta d\rho \leq \frac{m^2}{2} \log \left(\frac{1+r}{1-r} \right) + (1 - m^2)r. \quad \dots(16)$$

Inequality (15) coincides with the one given by Noor (1981), whereas inequality (16) shows an improvement in the result given in (Noor & Noor, 1992) for the case $\rho = 0$ therein.

Lemma 2.3. (Raina et al., 2020) Let $p \in \mathcal{P}^\lambda[\mathcal{A}, \mathcal{B}]$.

Then

$$\left(\frac{1-\mathcal{A}r}{1-\mathcal{B}r} \right)^\lambda \leq |p(z)| \leq \left(\frac{1+\mathcal{A}r}{1+\mathcal{B}r} \right)^\lambda \quad \mathcal{B} \neq 0, \quad \dots(17)$$

$$(1-\mathcal{A}r)^\lambda \leq |p(z)| \leq (1+\mathcal{A}r)^\lambda \quad \mathcal{B} = 0. \quad \dots(18)$$

Lemma 2.4. Let $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$, where $m \geq 2$ and $-1 \leq \mathcal{B} < \mathcal{A} \leq 1$. Then

$$|\arg f'(z)| \leq \begin{cases} \frac{m}{2} \frac{\mathcal{A}-\mathcal{B}}{\mathcal{B}} \arcsin(\mathcal{B}r), & \text{if } \mathcal{B} \neq 0 \\ \frac{m}{2} \mathcal{A}r, & \mathcal{B} = 0 \end{cases} \quad \dots(19)$$

Proof. Using the representation for $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$ in (Noor, 1994), we write

$$f'(z) = \frac{(f_1'(z))^{\frac{m+2}{4}}}{(f_2'(z))^{\frac{m-2}{4}}}, \quad \dots(20)$$

for some $f_1(z), f_2(z) \in \mathcal{V}_2[\mathcal{A}, \mathcal{B}]$. It is known in (Silvia, 1983) that if $f \in \mathcal{V}_2[\mathcal{A}, \mathcal{B}]$, then

$$|\arg f'(z)| \leq \begin{cases} \frac{\mathcal{A}-\mathcal{B}}{\mathcal{B}} \arcsin(\mathcal{B}r), & \text{if } \mathcal{B} \neq 0 \\ \mathcal{A}r, & \mathcal{B} = 0. \end{cases} \quad \dots(21)$$

Using this result and (20), the proof is completed.

Lemma 2.5. (Noor, 1994) Let $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$. Then

$$\frac{(1-\mathcal{B}r)^{q_1}}{(1+\mathcal{B}r)^{q_2}} \leq |f'(z)| \leq \frac{(1+\mathcal{B}r)^{q_1}}{(1-\mathcal{B}r)^{q_2}}, \quad \mathcal{B} \neq 0, \\ e^{-\frac{\mathcal{A}mr}{2}} \leq |f'(z)| \leq e^{\frac{\mathcal{A}mr}{2}}, \quad \mathcal{B} = 0,$$

where

$$q_1 = \left(\frac{\mathcal{A}-\mathcal{B}}{\mathcal{B}}\right) \left(\frac{m+2}{4}\right) \quad \text{and} \quad q_2 = \left(\frac{\mathcal{A}-\mathcal{B}}{\mathcal{B}}\right) \left(\frac{m-2}{4}\right). \quad \dots(22)$$

Throughout this work, unless otherwise stated, we assume $-1 \leq \mathcal{B} < \mathcal{A} \leq 1$, $-1 \leq \mathcal{D} < \mathcal{C} \leq 1$ and $m, m_1, m_2 \geq 2$.

RESULTS AND DISCUSSION

In this section, we shall proceed with the statements and proofs of our main results.

Theorem 3.1. Let $A(r, f)$ be the area bounded by the image curve $|z| = r$, $r \in (0, 1)$ under the function $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$. Then

$$L(r, f) \leq \begin{cases} 2 \left[\frac{\pi A(r, f)}{r} \left(r + \frac{m^2(\mathcal{B}-\mathcal{A})^2}{8\mathcal{B}^3} \left(\log \left(\frac{1+\mathcal{B}r}{1-\mathcal{B}r} \right) - 2\mathcal{B}r \right) \right) \right]^{\frac{1}{2}}, & \mathcal{B} \neq 0 \\ 2 \left[\frac{\pi A(r, f)}{r} \left(r + \frac{m^2 \mathcal{A}^2 r^3}{12} \right) \right]^{\frac{1}{2}}, & \mathcal{B} = 0. \end{cases}$$

Proof. Since $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$, then

$$\frac{(zf'(z))'}{f'(z)} = p(z), \quad \text{where } p \in \mathcal{P}_m[\mathcal{A}, \mathcal{B}]. \quad \dots(23)$$

By the definition of arc length, we have

$$L(r, f) = \int_0^{2\pi} |zf'(z)| d\theta \\ = \int_0^r \int_0^{2\pi} |(zf'(z))'| d\theta d\rho \\ = \int_0^r \int_0^{2\pi} |f'(z)| \left| \frac{(zf'(z))'}{f'(z)} \right| d\theta d\rho \\ \leq \left(\int_0^r \int_0^{2\pi} |f'(z)|^2 d\theta d\rho \right)^{\frac{1}{2}} \times \left(\int_0^r \int_0^{2\pi} |p(z)|^2 d\theta d\rho \right)^{\frac{1}{2}} \\ \leq \left(2\pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n-1} \right)^{\frac{1}{2}} \left(\int_0^r \int_0^{2\pi} |p(z)|^2 d\theta d\rho \right)^{\frac{1}{2}}. \quad \dots(24)$$

On applying Lemma 2.2, and using the fact that

$$A(r, f) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \quad a_1 = 1,$$

we obtain the required result.

As a consequence of the Theorem 3.1, we have Corollary 3.1. Let $f \in \mathcal{V}_m$. Then

$$L(r, f) \leq 2 \left[\frac{\pi A(r, f)}{r} \left(\frac{m^2}{2} \log \left(\frac{1+r}{1-r} \right) + (1-m^2) \right) \right]^{\frac{1}{2}}$$

or equivalently as

$$L(r, f) \leq m \left[\pi A(r, f) \log \left(\frac{1}{1-r} \right) \right]^{\frac{1}{2}} \quad \text{as } r \rightarrow 1$$

and for $f \in \mathcal{V}_2$, we have

$$L(r, f) = O \left[\pi A(r, f) \log \left(\frac{1}{1-r} \right) \right]^{\frac{1}{2}} \quad \text{as } r \rightarrow 1,$$

which was the result obtained by Nunokawa (1969) and O is a constant.

Theorem 3.2. If $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$, then so is $F'(z) = (-f'(z) \cdot f'(-z))^{\frac{1}{2}}$.

Proof. Suppose $f \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$. Then in (Noor, 1991a), we have the representation

$$f'(z) = \exp \left\{ \frac{\mathcal{A}-\mathcal{B}}{2\mathcal{B}} \int_{-\pi}^{\pi} \log(1 + \mathcal{B}ze^{it}) d\mu(t) \right\}, \quad \mathcal{B} \neq 0 \quad \dots(25)$$

and

$$f'(z) = \exp \left\{ \frac{\mathcal{A}}{2} \int_{-\pi}^{\pi} z e^{it} d\mu(t) \right\}, \quad \mathcal{B} = 0, \quad \dots(26)$$

where μ is a real valued of bounded variation on $[-\pi, \pi]$ satisfying the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2 \quad \text{and} \quad \int_{-\pi}^{\pi} |d\mu(t)| \leq m.$$

There fore for $\mathcal{B} \neq 0$,

$$F'(z) = -\exp \left\{ \frac{\mathcal{A} - \mathcal{B}}{4\mathcal{B}} \int_{-\pi}^{\pi} \log(1 + \mathcal{B}ze^{it}) d\mu(t) \right\} \\ \times \exp \left\{ \frac{\mathcal{A} - \mathcal{B}}{4\mathcal{B}} \int_{-\pi}^{\pi} \log(1 - \mathcal{B}ze^{it}) d\mu(t) \right\}.$$

Thus,

$$\frac{(zF'(z))'}{F'(z)} = 1 + \frac{\mathcal{A} - \mathcal{B}}{4} \int_{-\pi}^{\pi} \frac{ze^{-it}}{1 + \mathcal{B}ze^{it}} d\mu(t) - \frac{\mathcal{A} - \mathcal{B}}{4} \int_{-\pi}^{\pi} \frac{ze^{-it}}{1 - \mathcal{B}ze^{it}} d\mu(t) \\ = \frac{1}{2} \left\{ \frac{1}{2} \int_{-\pi}^{\pi} \frac{1 + \mathcal{A}ze^{-it}}{1 + \mathcal{B}ze^{it}} d\mu(t) + \frac{1}{2} \int_{-\pi}^{\pi} \frac{1 - \mathcal{A}ze^{-it}}{1 - \mathcal{B}ze^{it}} d\mu(t) \right\} \\ = \frac{1}{2} (p(z) + p(-z)), \quad \text{where } p(z) \in \mathcal{P}_m[\mathcal{A}, \mathcal{B}].$$

In view of the convexity property of the class $\mathcal{P}_m[\mathcal{A}, \mathcal{B}]$, it follows that $F \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$, which is the required result. The case $\mathcal{B} = 0$ also follows the same arguments.

Theorem 3.3. Let $f, g \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$ and suppose that for any $\mathcal{A}$ and $\mathcal{B}$ satisfying the condition $3\mathcal{A}\mathcal{B} + \mathcal{A}^2 + \mathcal{B}^2 < 0$, $F(z) = \beta f(z) + (1 - \beta)g(z) \in \mathcal{V}_m[1, -1]$ ($\beta \in \mathbb{C} - \{0, 1\}$) if $|z| < r_0$, wherer r_0 is the smallest positive root satisfying the equation:

$$\begin{cases} (1 - \mathcal{A}\mathcal{B}r^2) \cos\left(\frac{\gamma}{2} + \frac{m}{2} \left(\frac{\mathcal{A} - \mathcal{B}}{\mathcal{B}}\right) \arcsin(\mathcal{B}r)\right) - (\mathcal{A} - \mathcal{B})r = 0, \\ \cos\left(\frac{\gamma}{2} + \frac{\mathcal{A}mr}{2}\right) - \mathcal{A}r, \end{cases} \quad \begin{matrix} \mathcal{B} \neq 0 \\ \mathcal{B} = 0, \end{matrix} \quad \dots(27)$$

where $0 \leq \gamma = \arg\left(\frac{\beta}{1 - \beta}\right) < \pi$.

Proof. Since $f, g \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$, then there exist $p_1, p_2, h_1, h_2 \in \mathcal{P}[\mathcal{A}, \mathcal{B}]$ such that

$$\frac{(zf'(z))'}{f(z)} = \frac{m+2}{4}p_1(z) - \frac{m-2}{4}p_2(z) \quad \dots(28)$$

$$\frac{(zf'(z))'}{f(z)} = \frac{m+2}{4}p_1(z) - \frac{m-2}{4}p_2(z) \quad \dots(29)$$

By direct calculation,

$$\frac{(zF'(z))'}{F(z)} = \frac{(zf'(z))'}{f(z)} \left[1 + \left(\frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right)^{-1} \right]^{-1} \\ + \frac{(zg'(z))'}{g(z)} \left[1 + \frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right]^{-1} \quad \dots(30)$$

$$= \frac{m+2}{4} \left\{ \left[1 + \frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right]^{-1} h_1(z) \right. \\ \left. + \left[1 + \left(\frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right)^{-1} \right]^{-1} p_1(z) \right\} \\ - \frac{m-2}{4} \left\{ \left[1 + \frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right]^{-1} h_2(z) \right. \\ \left. + \left[1 + \left(\frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right)^{-1} \right]^{-1} p_2(z) \right\} \\ = \frac{m+2}{4}w_1(z) - \frac{m-2}{4}w_2(z), \quad \dots(31)$$

where we have used (28) and (29), and

$$w_i = \frac{u_i}{1 + \lambda e^{i\alpha}} + \frac{v_i}{1 + \lambda^{-1}e^{-i\alpha}}, \quad i = 1, 2, \quad \dots(32)$$

with $u_i = h_i$, $v_i = p_i$ ($i = 1, 2$) and

$$\lambda = \left| \frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)} \right|, \quad \alpha = \arg\left(\frac{\beta}{1 - \beta} \frac{f'(z)}{g'(z)}\right).$$

In view of Lemma 2.4, we have

$$|\alpha| \leq \gamma + m \frac{\mathcal{A} - \mathcal{B}}{\mathcal{B}} \arcsin(\mathcal{B}r)$$

and

$$\sec \frac{\alpha}{2} \leq \sec \left(\frac{\gamma}{2} + \frac{m}{2} \frac{\mathcal{A} - \mathcal{B}}{\mathcal{B}} \arcsin(\mathcal{B}r) \right).$$

Applying Lemma 2.1 to (32), we obtain

$$\text{Rew}_i \geq a - d \sec \frac{\alpha}{2},$$

where

$$a = \frac{1 - \mathcal{A}\mathcal{B}r^2}{1 - \mathcal{B}^2r^2}, \quad d = \frac{(\mathcal{A} - \mathcal{B})r}{1 - \mathcal{B}^2r^2} \quad \dots(33)$$

and $a > b$ holds for any $\mathcal{A}$ and $\mathcal{B}$ satisfying the condition $3\mathcal{A}\mathcal{B} + \mathcal{A}^2 + \mathcal{B}^2 < 0$. Thus, the right side of (33) is positive if

$$0 < (1 - \mathcal{A}\mathcal{B}r^2) \cos\left(\frac{\gamma}{2} + \frac{m}{2} \frac{(\mathcal{A} - \mathcal{B})}{\mathcal{B}} \arcsin \mathcal{B}r\right) - (\mathcal{A} - \mathcal{B})r.$$

Let

$$\mathcal{T}(r) = (1 - \mathcal{A}\mathcal{B}r^2) \cos\left(\frac{\gamma}{2} + \frac{m}{2} \frac{(\mathcal{A} - \mathcal{B})}{\mathcal{B}} \arcsin \mathcal{B}r\right) - (\mathcal{A} - \mathcal{B})r.$$

Then

$$\mathcal{T}(0) = \cos \frac{\gamma}{2} > 0 \quad \text{and}$$

$$\mathcal{T}\left(\frac{1}{\mathcal{B}} \sin\left(\frac{\mathcal{B}(\pi - \gamma)}{m(\mathcal{A} - \mathcal{B})}\right)\right) = -\frac{(\mathcal{A} - \mathcal{B})}{\mathcal{B}} \sin\left(\frac{\mathcal{B}(\pi - \gamma)}{m(\mathcal{A} - \mathcal{B})}\right) < 0.$$

This means that $\mathcal{T}(r)$ has a solution in the interval $\left[0, \frac{1}{\mathcal{B}} \sin\left(\frac{\mathcal{B}(\pi - \gamma)}{m(\mathcal{A} - \mathcal{B})}\right)\right]$. Hence from (32), we conclude that $F \in \mathcal{V}_m[1, -1]$.

For $\mathcal{A} = 1$, $\mathcal{B} = -1$, $m = 2$ in Theorem 3.3, we have the following:

Corollary 3.2. (Stump, 1971) Let $f, g \in \mathcal{V}_2[1, -1]$. Then $F(z) = \beta f(z) + (1 - \beta)g(z)$ is convex in the disc $|z| < r_c$, and r_c is the smallest positive root of the equation:

$$(1 + r^2) \cos\left(\frac{\gamma}{2} + 2 \arcsin r\right) - 2r = 0, \quad \dots(34)$$

where $0 \leq \gamma = \arg\left(\frac{\beta}{1 - \beta}\right) < \pi$.

Theorem 3.4. Let $f, g \in \mathcal{CCV}_{m_1 m_2}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ and $0 \leq \gamma = \arg\left(\frac{\beta}{1 - \beta}\right) < \pi$ and suppose for any $\mathcal{C}$ and $\mathcal{D}$ satisfying the condition $3\mathcal{C}\mathcal{D} + \mathcal{C}^2 + \mathcal{D}^2 < 0$, $F(z) = \beta f(z) + (1 - \beta)g(z) \in \mathcal{CCV}_{m_1 m_2}[1, -1, 1, -1]$ in the disk $|z| < \min\{r_1, r_2\}$, where r_1 with $m = m_1$ and r_2 is the least roots of the equation:

$$\begin{cases} (1 - \mathcal{C}\mathcal{D}r^2) \cos\left(\frac{\gamma}{2} + \frac{m_2}{2} \left(\frac{\mathcal{C} - \mathcal{D}}{\mathcal{D}}\right) \arcsin(\mathcal{D}r)\right) - (\mathcal{C} - \mathcal{D})r = 0, \\ \cos\left(\frac{\gamma}{2} + \frac{\mathcal{C}m_2r}{2}\right) - \mathcal{C}r, \end{cases} \quad \begin{matrix} \mathcal{D} \neq 0 \\ \mathcal{D} = 0. \end{matrix} \quad \dots(35)$$

Proof. Let $f, g \in \mathcal{CCV}_{m_1 m_2}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$. Then there exist $h(z), q(z) \in \mathcal{V}_{m_1}[\mathcal{A}, \mathcal{B}]$ such that

$$\frac{f'(z)}{h'(z)}, \frac{g'(z)}{q'(z)} \in \mathcal{P}_{m_2}[\mathcal{C}, \mathcal{D}]. \quad \dots(36)$$

Suppose $\mathcal{D} \neq 0$ and let $G(z) = \beta h(z) + (1 - \beta)q(z)$. Then by Theorem 3.3, $G \in \mathcal{V}_{m_1}[1, -1]$ in the disk $|z| < r_1$.

Therefore by simple calculation, it follows that

$$\begin{aligned} \frac{F'(z)}{G'(z)} &= \frac{f'(z)}{h'(z)} \left[1 + \frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right]^{-1} \\ &\quad + \frac{g'(z)}{q'(z)} \left[1 + \left(\frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right)^{-1}\right]^{-1}. \end{aligned} \quad \dots(37)$$

From (36), it implies there exist $h_3, h_4, p_3, p_4 \in \mathcal{P}[\mathcal{C}, \mathcal{D}]$ such that (37) becomes

$$\begin{aligned} \frac{F'(z)}{G'(z)} &= \frac{m_2 + 2}{4} \left\{ \left[1 + \frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right]^{-1} h_3(z) \right. \\ &\quad \left. + \left[1 + \left(\frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right)^{-1}\right]^{-1} p_3(z) \right\} \\ &\quad - \frac{m_2 - 2}{4} \left\{ \left[1 + \frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right]^{-1} h_4(z) \right. \\ &\quad \left. + \left[1 + \left(\frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right)^{-1}\right]^{-1} p_4(z) \right\} \\ &= \frac{m + 2}{4} w_1(z) - \frac{m - 2}{4} w_2(z), \end{aligned}$$

where

$$w_i = \frac{u_i}{1 + \lambda e^{i\alpha}} + \frac{v_i}{1 + \lambda^{-1} e^{-i\alpha}}, \quad i = 3, 4,$$

with $u_i = h_i$, $v_i = p_i$ ($i = 3, 4$) and

$$\lambda = \left| \frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)} \right|, \quad \alpha = \arg\left(\frac{\beta}{1 - \beta} \frac{h'(z)}{q'(z)}\right).$$

Using Lemma 2.1 with

$$a = \frac{1 - \mathcal{C}\mathcal{D}r^2}{1 - \mathcal{D}^2r^2}, \quad d = \frac{(\mathcal{C} - \mathcal{D})r}{1 - \mathcal{D}^2r^2} \quad \dots(38)$$

and proceeding in the same manner as in Theorem 3.3, we obtain the required result.

Corollary 3.3. Let $f, g \in \mathcal{CCV}_2[1, -1]$. Then $F(z) = \beta f(z) + (1 - \beta)g(z)$ is close-to-convex in the disc $|z| < r_c$, and r_c is the smallest positive root of (34).

Theorem 3.5. Let $f, g \in \mathcal{QCV}_{m_1 m_2}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ and $0 \leq \gamma = \arg\left(\frac{\beta}{1-\beta}\right) < \pi$ and suppose for any $\mathcal{C}$ and $\mathcal{D}$ satisfying the condition $3\mathcal{C}\mathcal{D} + \mathcal{C}^2 + \mathcal{D}^2 < 0$,

$F(z) = \beta f(z) + (1 - \beta)g(z) \in \mathcal{QCV}_{m_1 m_2}[1, -1, 1, -1]$ in the disk $|z| < \min\{r_1, r_2\}$, where r_1 is defined in Theorem 3.3 with $m = m_1$ and r_2 is the least roots of (35).

Proof. The proof follows the same setting as in Theorem 3.4.

Corollary 3.4. Let $f, g \in \mathcal{QCV}_2[1, -1]$. Then $F(z) = \beta f(z) + (1 - \beta)g(z)$ is quasi convex in the disc $|z| < r_c$, and r_c is the smallest positive root of (34).

Using Kaplan(1952)method, we prove Theorem 3.6 and the rest of the proofs associated with corresponding theorems in this presentation follow the techniques of Noor (1983).

Theorem 3.6. Let $f \in K_m^\lambda[\mathcal{A}, \mathcal{B}]$. Then for $z = re^{i\theta}$ $\theta_1 < \theta_2$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\pi \left(\lambda + \frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \left(\frac{m}{2} - 1 \right) \right).$$

Proof $\frac{f'(z)}{g'(z)} \in \mathcal{P}^\lambda[\mathcal{A}, \mathcal{B}] \Rightarrow zf'(z) = zg'(z)p^\lambda(z)$ for some $g \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$ and $p \in \mathcal{P}[\mathcal{A}, \mathcal{B}]$. Let $p(z) = \arg zf'(z)$ and $q(z) = \arg zg'(z)$. Then by definition

$$|p(z) - q(z)| < \frac{\lambda\pi}{2} \quad \dots(39)$$

Let $P(r, \theta) = p(re^{i\theta}) + \theta$ and $Q(r, \theta) = q(re^{i\theta}) + \theta$ be defined for all $r \in [0, 1)$ and $\theta \in [0, 2\pi]$. Then

$$|P(r, \theta_2) - P(r, \theta_1)| \leq |P(r, \theta_2) - Q(r, \theta_2)| + |Q(r, \theta_2) - Q(r, \theta_1)| + |Q(r, \theta_1) - P(r, \theta_1)|. \quad \dots(40)$$

It is easy to see that for $g \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]$,

$$g'(z) = (g'_1)^{\frac{\mathcal{A}-\mathcal{B}}{1-\mathcal{B}}}, \quad g_1 \in \mathcal{V}_m. \quad \dots(41)$$

Since every function $g_1 \in \mathcal{V}_m$ is a close-to-convex function of order $\frac{m}{2} - 1$ (Brannan, 1969), then

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zg'_1(z))'}{g'_1(z)} d\theta > -\pi \left(\frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \right) \left(\frac{m}{2} - 1 \right),$$

i.e.,

$$|Q(r, \theta_2) - Q(r, \theta_1)| < \pi \left(\frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \right) \left(\frac{m}{2} - 1 \right). \quad \dots(42)$$

In view of (39), (40) and (42), we obtain

$$|Q(r, \theta_2) - Q(r, \theta_1)| < \lambda + \pi \left(\frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \right) \left(\frac{m}{2} - 1 \right). \quad \dots(43)$$

Hence, for $\theta_1 < \theta_2$ and $z = re^{i\theta}$, we obtain the result.

Remark 3.1.

(i) From Theorem 3.6, it is obvious that $f \in K_m^\lambda[\mathcal{A}, \mathcal{B}]$ is a close-to-convex function and hence univalent for $2 \leq m \leq 2 \left[1 + \left(\frac{1-\mathcal{B}}{\mathcal{A}-\mathcal{B}} \right) (1 - \lambda) \right]$.

(ii) Goodman (1971) defines the class $\mathcal{K}(\beta)$ of normalized analytic functions f which are close-to-convex functions of order $\beta \geq 0$ as follows: A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{K}(\beta)$ if f is analytic in $\mathcal{U}$, $f'(z) \neq 0$ and for $z = re^{i\theta}$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $\beta \geq 0$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\pi\beta.$$

We note that

$$K_m^\lambda[\mathcal{A}, \mathcal{B}] \subset \mathcal{K} \left(\lambda + \frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \left(\frac{m}{2} - 1 \right) \right)$$

and the functions in $K_m^\lambda[\mathcal{A}, \mathcal{B}]$ need not to be finitely-valent for

$$m > 2 \left[1 + \left(\frac{1-\mathcal{B}}{\mathcal{A}-\mathcal{B}} \right) (1 - \lambda) \right].$$

From Theorem 3.6, we have the following covering theorem.

Theorem 3.7. Let $2 \leq m \leq 2 \left[1 + \left(\frac{1-\mathcal{B}}{\mathcal{A}-\mathcal{B}} \right) (1 - \lambda) \right]$. Then the range of U under functions $f \in K_m^\lambda[\mathcal{A}, \mathcal{B}]$ contains the disc

$$|z| < \frac{2}{\lambda(\mathcal{A} - \mathcal{B}) + m \left(\frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \right) + 4}.$$

Proof. Since $f \in K_m^\lambda[\mathcal{A}, \mathcal{B}]$, then from (41), we have for $z \in \mathcal{U}$,

$$f'(z) = (g'_1(z))^{1-\alpha} p^\lambda(z), \quad f \in \mathcal{V}_m, \quad p \in \mathcal{P}[\mathcal{A}, \mathcal{B}], \quad \alpha = \frac{1-\mathcal{A}}{1-\mathcal{B}}. \quad \dots(44)$$

Let $f(z)$ be given by (1), $g_1(z) = z + b_2 z^2 + b_3 z^3 + \dots$ and $p(z) = 1 + c_1 z + c_2 z^2 + \dots$. Then by (44), we obtain

$$1 + 2a_2 z + 3a_3 z^2 + \dots = \left(1 + 2b_2 z + 3b_3 z^2 + \dots\right)^{1-\alpha} \times \left(1 + c_1 z + c_2 z^2 + \dots\right)^\lambda$$

and equating coefficients of z , we obtain $2a_2 = \lambda c_1 + 2\alpha b_2$. Therefore,

$$|a_2| \leq \frac{1}{2} (\lambda |c| + 2(1-\alpha)|b_2|) \quad \dots(45)$$

$$\leq \frac{1}{2} \left[\lambda(\mathcal{A} - \mathcal{B}) + m \left(\frac{\mathcal{A} - \mathcal{B}}{1 - \mathcal{B}} \right) \right], \quad \dots(46)$$

where we have used the bounds for functions in $\mathcal{P}[\mathcal{A}, \mathcal{B}]$ (Saliu, 2019) and in $\mathcal{V}_m$ (Lehto, 1952).

The Koebe one-quarter theorem states that each omitted value w of the univalent function $f(z)$ of the form (1) satisfies

$$|w| > \frac{1}{2 + |a_2|}. \quad \dots(47)$$

In view of (45) and (47), we have the required result. To establish the next theorem, we need the following hypergeometric functions defined as follow:

If $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, then

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt$$

and

$${}_1F_1(b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} e^{tz} dt.$$

Theorem 3.8. Let $f(z) \in K_m^\lambda[\mathcal{A}, \mathcal{B}]$. Then for $\mathcal{B} \leq 0$,

$$\Phi_2(\mathcal{A}, \mathcal{B}, m, r) \leq |f(z)| \leq \Phi_1(\mathcal{A}, \mathcal{B}, m, r),$$

where

$$\Phi_1(\mathcal{A}, \mathcal{B}, m, r) = \begin{cases} \frac{1}{b\mathcal{B}} \sum_{k=0}^{\infty} \binom{\lambda}{k} \left(\frac{\mathcal{A}}{\mathcal{B}}\right)^{\lambda-k} \left(\frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}}\right)^k 2^{\gamma_k+1} [{}_2F_1(a, b; c; -1) - r_1^{1-q_2} {}_2F_1(a, b; c; -r_1)], & \mathcal{B} \neq 0, \\ \sum_{k=0}^{\infty} \binom{\lambda}{k} \frac{\mathcal{A}^k r^{k+1}}{k} {}_1F_1\left(k+1, k+2; \frac{\mathcal{A}m}{2}\right), & \mathcal{B} = 0, \end{cases}$$

$$\Phi_2(\mathcal{A}, \mathcal{B}, m, r) = \begin{cases} \frac{1}{b\mathcal{B}} \sum_{k=0}^{\infty} \binom{\lambda}{k} \left(\frac{\mathcal{A}}{\mathcal{B}}\right)^{\lambda-k} \left(\frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}}\right)^k 2^{\gamma_k+1} [r_2^{1-q_2} {}_2F_1(a, b; c; -r_2) - {}_2F_1(a, b; c; -1)], & \mathcal{B} \neq 0, \\ \sum_{k=0}^{\infty} (-1)^k \binom{\lambda}{k} \frac{\mathcal{A}^k r^{k+1}}{k} {}_1F_1\left(k+1, k+2; \frac{-\mathcal{A}m}{2}\right), & \mathcal{B} = 0, \end{cases}$$

q_2 is being given by (27)

$$\gamma_k = \left(\frac{\mathcal{A}-\mathcal{B}}{\mathcal{B}} - k\right), \quad a = \gamma + 2, \quad b = 1 - q_2, \quad r_1 = \frac{1 - \mathcal{B}r}{1 + \mathcal{B}r},$$

$$r_2 = r_1^{-1} \text{ and } c = b + 1. \quad \dots(49)$$

$$f(z) = \int_0^z \frac{(1 + \mathcal{B}x\xi)^{q_1}}{(1 - \mathcal{B}x\xi)^{q_2}} \left(\frac{1 + \mathcal{A}x\xi}{1 + \mathcal{B}x\xi}\right)^\lambda d\xi, \quad |x| = 1, \quad \dots(50)$$

The inequality (48) cannot be improved because the function

attains the equality.

Proof. Let $z = re^{i\theta}$. Then for $f \in \mathcal{K}_m^\lambda[\mathcal{A}, \mathcal{B}]$,

$$|f(z)| \leq \int_0^r |g(\rho e^{i\theta})| |p(\rho e^{i\theta})| d\rho, \quad p \in \mathcal{P}^\lambda[\mathcal{A}, \mathcal{B}], \quad g \in \mathcal{V}_m[\mathcal{A}, \mathcal{B}]. \quad \dots(51)$$

In view of Lemma 2.3, Lemma 2.5 and binomial expansion we obtain

$$\begin{aligned} |f(z)| &\leq \int_0^r \frac{(1 + \mathcal{B}\rho)^{q_1}}{(1 - \mathcal{B}\rho)^{q_2}} \left(\frac{1 + \mathcal{A}\rho}{1 + \mathcal{B}\rho} \right)^\lambda d\rho \\ &= \sum_{k=0}^{\infty} \binom{\lambda}{k} \left(\frac{\mathcal{A}}{\mathcal{B}} \right)^{\lambda-k} \left(\frac{\mathcal{B} - \mathcal{A}}{\mathcal{B}} \right)^k \int_0^r \frac{(1 + \mathcal{B}\rho)^{q_1}}{(1 - \mathcal{B}\rho)^{q_2}} \frac{d\rho}{(1 + \mathcal{B}\rho)^k} \\ &= \sum_{k=0}^{\infty} \binom{\lambda}{k} \left(\frac{\mathcal{A}}{\mathcal{B}} \right)^{\lambda-k} \left(\frac{\mathcal{B} - \mathcal{A}}{\mathcal{B}} \right)^k I, \end{aligned} \quad \dots(52)$$

where

$$I = \int_0^r \left(\frac{1 + \mathcal{B}\rho}{1 - \mathcal{B}\rho} \right)^{q_2} (1 + \mathcal{B}\rho)^\gamma d\rho. \quad \dots(53)$$

Using the transformation $u = \frac{1 + \mathcal{B}\rho}{1 - \mathcal{B}\rho}$, we have that

$$\begin{aligned} I &= -\frac{2^{\gamma+1}}{\mathcal{B}} \int_1^{r_1} u^{-q_2} (1 + u)^{-(\gamma+2)} du \\ &= \frac{2^{\gamma+1}}{\mathcal{B}} [I_1 - I_2], \end{aligned} \quad \dots(54)$$

where

$$I_1 = \int_0^1 u^{-q_2} (1 + u)^{-(\gamma+2)} du \quad \dots(55)$$

$$= \frac{1}{b} {}_2F_1(a, b; c; -1), \quad \dots(56)$$

with

$$\begin{aligned} a &= \gamma + 2, \\ b &= 1 - q_2 > 0 \text{ since } \mathcal{B} \leq 0, \\ c &= 2 - q_2 = b + 1, \end{aligned}$$

and

$$I_2 = \int_0^{r_1} u^{-q_2} (1 + u)^{-(\gamma+2)} du.$$

Let $u = r_1 v$. Then

$$\begin{aligned} I_2 &= r_1^{1-q_2} \int_0^{r_1} v^{-q_2} (1 + v)^{-(\gamma+2)} dv \\ &= \frac{r_1^{1-q_2}}{b} {}_2F_1(a, b; c; -r_1). \end{aligned} \quad \dots(57)$$

Using (57), (55) in (54) and (54) in (52), we have

$$\begin{aligned} |f(z)| &\leq \frac{1}{b\mathcal{B}} \sum_{k=0}^{\infty} \binom{\lambda}{k} \left(\frac{\mathcal{A}}{\mathcal{B}} \right)^{\lambda-k} \left(\frac{\mathcal{B} - \mathcal{A}}{\mathcal{B}} \right)^k 2^{\gamma+1} \\ &\quad \left[{}_2F_1(a, b; c; -1) - r_1^{1-q_2} {}_2F_1(a, b; c; -r_1) \right]. \end{aligned} \quad \dots(58)$$

For the case $\mathcal{B} = 0$, we have from (51), Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned} |f(z)| &\leq \int_0^r e^{\frac{\mathcal{A}m\rho}{2}} (1 + \mathcal{A}\rho)^\lambda d\rho \\ &= \sum_{k=0}^{\infty} \binom{\lambda}{k} \mathcal{A}^k \int_0^r e^{\frac{\mathcal{A}m\rho}{2}} \rho^k d\rho \\ &= \sum_{k=0}^{\infty} \binom{\lambda}{k} \mathcal{A}^k r^{k+1} \int_0^1 u^k e^{\frac{\mathcal{A}mr}{2}u} du, \quad \text{by putting } \rho = ur \\ &= \sum_{k=0}^{\infty} \binom{\lambda}{k} \frac{\mathcal{A}^k r^{k+1}}{k} {}_1F_1\left(k+1, k+2; \frac{-\mathcal{A}mr}{2}\right). \end{aligned} \quad \dots(59)$$

To prove the lower bound of (48), we consider a point z_0 ($|z_0| = r < 1$) such that $|f(z)| \geq |f(z_0)|$ ($\forall z: |z| = r$). Let C be an arc in U which is mapped by the function $w = f(z)$ onto a line segment L connecting origin to the point $f(z_0)$ and lying completely in the image of U under f . Thus, by Lemma 2.3 and Lemma 2.5, we get

$$\begin{aligned} |f(z)| &\geq |f(z_0)| = \int_L |dw| \\ &= \int_C |f'(z)| |dz| \geq \int_0^r \frac{(1 - \mathcal{B}\rho)^{q_1}}{(1 + \mathcal{B}\rho)^{q_2}} \left(\frac{1 - \mathcal{A}\rho}{1 - \mathcal{B}\rho} \right)^\lambda d\rho. \end{aligned} \quad \dots(60)$$

Employing the same techniques used for finding the upper bound in (3.30), we obtain the lower bound. This completes the proof.

For $\lambda = 1$, in Theorem 3.8, we obtain the following corollary:

Corollary 3.5. Let $f(z) \in \mathcal{K}_m^1[\mathcal{A}, \mathcal{B}]$. Then for $\mathcal{B} \leq 0$, $\Psi_2(\mathcal{A}, \mathcal{B}, m, r) \leq |f(z)| \leq \Psi_1(\mathcal{A}, \mathcal{B}, m, r)$,

where

$$\Psi_1(\mathcal{A}, \mathcal{B}, m, r) = \begin{cases} \frac{2^{\gamma_1+1}}{b\mathcal{B}^2} \left\{ \left[2\mathcal{A}_2 F_1(\gamma_1 + 3, b; c; -1) + (\mathcal{B} - \mathcal{A}) {}_2F_1(\gamma_1 + 2, b; c; -1) \right] \right. \\ \left. - r_1^{1-q_2} \left[2\mathcal{A}_2 F_1(\gamma_1 + 3, b; c; -r_1) + (\mathcal{B} - \mathcal{A}) {}_2F_1(\gamma_1 + 2, b; c; -r_1) \right] \right\}, & \mathcal{B} \neq 0, \\ \frac{2}{\mathcal{A}m^2} \left\{ e^{\frac{\mathcal{A}mr}{2}} [(1 + \mathcal{A}r)m - 2] + 2 - m \right\}, & \mathcal{B} = 0 \end{cases}$$

and

$$\Psi_2(\mathcal{A}, \mathcal{B}, m, r) = \begin{cases} \frac{2^{\gamma_1+1}}{b\mathcal{B}^2} \left\{ r_2^{1-q_2} \left[2\mathcal{A}_2 F_1(\gamma_1 + 3, b; c; -r_2) + (\mathcal{B} - \mathcal{A}) {}_2F_1(\gamma_1 + 2, b; c; -r_2) \right] \right. \\ \left. - \left[2\mathcal{A}_2 F_1(\gamma_1 + 3, b; c; -1) + (\mathcal{B} - \mathcal{A}) {}_2F_1(\gamma_1 + 2, b; c; -1) \right] \right\}, & \mathcal{B} \neq 0, \\ \frac{2}{\mathcal{A}m^2} \left\{ e^{-\frac{\mathcal{A}mr}{2}} [2 - (1 - \mathcal{A}r)m] + m - 2 \right\}, & \mathcal{B} = 0. \end{cases}$$

The inequality is sharp for the function defined by (50) with $\lambda=1$.

Proof. For $\mathcal{B} \neq 0$, and $\lambda = 1$, (52) becomes

$$\begin{aligned} |f(z)| &\leq \int_0^r \frac{(1 + \mathcal{B}\rho)^{q_1}}{(1 - \mathcal{B}\rho)^{q_2}} \left(\frac{1 + \mathcal{A}\rho}{1 + \mathcal{B}\rho} \right) d\rho \\ &= \frac{\mathcal{A}}{\mathcal{B}} \int_0^r \frac{(1 + \mathcal{B}\rho)^{q_1}}{(1 - \mathcal{B}\rho)^{q_2}} d\rho + \\ &\quad \left(1 - \frac{\mathcal{A}}{\mathcal{B}} \right) \int_0^r \frac{(1 + \mathcal{B}\rho)^{q_1}}{(1 - \mathcal{B}\rho)^{q_2}} (1 + \mathcal{B}\rho)^{-1} d\rho. \end{aligned} \quad \dots(61)$$

Evaluating the two integrals in (61) the same way we evaluated the integral (53), we easily obtain the upper bound. Applying the technique used to obtain (60) for $\lambda = 1$ and implementing the procedures to prove the upper bound in Corollary 3.5, then the lower bound is obvious.

In the case $\mathcal{B} = 0$ and $\lambda = 1$, then by Lemma 2.3 and Lemma 2.5, (52) reduces to

$$|f(z)| \leq \int_0^r e^{\frac{\mathcal{A}m\rho}{2}} (1 + \mathcal{A}\rho) d\rho, \quad \dots(62)$$

which gives the upper bound in this case, using integration by parts. In view of the method used to obtain (3.38), the lower bound is proved for $\mathcal{B} = 0$ and $\lambda = 1$.

For $\mathcal{B} = -1$ and $\mathcal{A} = 1$ in Theorem 3.8, our investigation reduces to the one obtained by Noor (2002) and contained in the following corollary:

Corollary 3.6. Let $f \in \mathcal{K}_m^\lambda$. Then

$$\begin{aligned} \frac{1}{m + 2\lambda} \left[\left(\frac{1-r}{1+r} \right)^{\frac{m}{2} + \lambda} - 1 \right] &\leq |f(z)| \leq \\ \frac{1}{m + 2\lambda} \left[\left(\frac{1+r}{1-r} \right)^{\frac{m}{2} + \lambda} - 1 \right]. \end{aligned} \quad \dots(63)$$

This result is sharp.

CONCLUSION

In this study, generalized close-to-convex, quasi-convex and strongly close-to-convex classes of functions were introduced. In addition, geometric properties of said functions, including the arc length result, radii problems, the necessary condition for univalence, covering theorem and distortion results were investigated. Many consequences that showed the validity of the present investigations were also highlighted.

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